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Steady three-dimensional water-wave patterns on a finite-depth fluid

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The formation of doubly-periodic patterns on the surface of a fluid layer with a uniform velocity field and constant depth is considered. The fluid is assumed to be inviscid and the flow irrotational. The problem of steady patterns is shown to have a novel variational formulation, which includes a new characterization of steady uniform mean flow, and steady uniform flow coupled with steady doubly periodic patterns. A central observation is that mean flow can be characterized geometrically by associating it with symmetries. The theory gives precise information about the role of the ten natural parameters in the problem which govern the wave–mean flow interaction for steady patterns in finite depth. The formulation is applied to the problem of interaction of capillary–gravity short-crested waves with oblique travelling waves, leading to several new observations for this class of waves. Moreover, by including oblique travelling waves and short-crested waves in the same analysis, new bifurcations of short-crested waves are found, which give rise to mixed waves which may have complicated spatial structure.

1. Introduction

Although great progress has been made in understanding two-dimensional water waves – particularly steady two-dimensional waves – there is an extraordinary number of open questions about three-dimensional ocean waves and patterns – with two horizontal and one vertical direction. In this paper we address several questions about *steady* three-dimensional water waves, particularly when the depth is finite, and mean flow effects become important. The fluid is assumed to be inviscid and the flow irrotational. In the two-dimensional case, steady periodic travelling waves can be viewed either as waves on a quiescent fluid that are moving but steady relative to a moving frame, or, the basic flow can be taken to be a uniform flow of depth h_0 and speed u_0, and then the wave can be considered as a stationary periodic wave on the uniform flow, in a fixed reference frame. It is this latter view that we generalize to two horizontal space dimensions. The main class of waves we consider are *doubly periodic* patterns on a finite-depth fluid, and our goals in this paper are fourfold: (*a*) to derive a variational principle for uniform flows with two horizontal directions which form a basic model for mean flow – indeed our theory gives a novel answer to the question: *what is mean flow?*, (*b*) to derive a variational principle for doubly periodic patterns

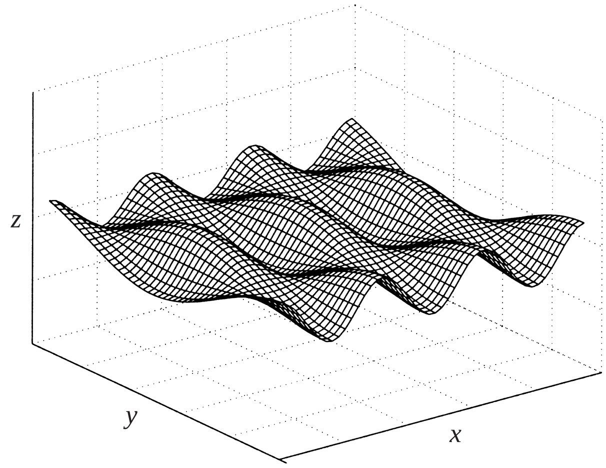

FIGURE 1. Schematic of a short-crested Stokes' wave propagating in the x-direction.

coupled with a mean flow, (*c*) to develop a theory which identifies all the parameters in the problem and their rôle – indeed our theory identifies ten parameters! – and (*d*) to apply this theory to the problem of short-crested Stokes waves on finite-depth water (including shallow water), where there are many open questions about the implications of mean flow for these waves.

The paradigm of the steady three-dimensional doubly periodic wave is the *short-crested Stokes wave*. This wave, which arises at the linear level from the superposition of two oblique travelling waves with the same wavenumber and the same amplitude, is a symmetric doubly periodic wave. A schematic of the linear superposition of two such linear waves is shown in figure 1. Special cases of course occur when both waves travel in opposite directions (the resulting wave is a standing wave) or when both waves travel in nearly the same direction.

Short-crested waves are of great interest in a range of maritime applications. They are known to affect sediment transport, appear in wave reflection by a sea-wall, and are relevant to remote sensing in the open ocean (cf. Silvester 1975, 1977; Hsu, Tsuchiya & Silvester 1979; Halliwell & Machen 1981; Silvester & Hsu 1997; Ioualalen *et al.* 1999). In particular, Ioualalen *et al.* (1999) showed how three-dimensionality could play a crucial role in altimetry by introducing a substantial bias to the sea-level and wind modulus measurements. Recent results on the theory of short-crested gravity Stokes waves and their linear stability include Hsu *et al.* (1979), who obtained a third-order approximation to short-crested waves by using a classical perturbation method, Roberts (1983) and Roberts & Schwartz (1983), who computed numerically short-crested waves in deep water, Roberts & Peregrine (1983), who considered the limit as both oblique waves become almost parallel, Marchant & Roberts (1987), who computed numerically short-crested waves in water of finite depth, Ioualalen & Kharif (1993, 1994), Badulin *et al.* (1995) and Ioualalen, Roberts & Kharif (1996), who studied the linear instability of short-crested waves. Experimental results on short-crested Stokes waves have recently been reported by Kimmoun, Branger & Kharif (1999). In addition to the papers already quoted, there is the first paper to our knowledge with rigorous results on three-dimensional waves: Reeder & Shinbrot (1981) proved the existence of small-amplitude short-crested waves in a certain region of parameter space. Sun (1993) provided an alternative construction of short-crested waves. In both papers, the proof works outside a forbidden set which is given in § 7 of

Reeder & Shinbrot (1981) (see also § 6 of the review article of Dias & Kharif 1999). Roughly speaking, the forbidden set consists of the parameters which allow resonances between harmonics. Longuet-Higgins & Phillips (1962) considered more general three-dimensional waves, resulting from the interaction of two weakly nonlinear waves in deep water, with differing wavenumbers and differing amplitudes. Hogan, Gruman & Stiassne (1988) generalized the work of Longuet-Higgins & Phillips (1962) by introducing surface tension. Other classes of steady three-dimensional water waves have been introduced by Saffman & Yuen (1980) (see also the review articles by Saffman & Yuen 1985 and Dias & Kharif 1999). The equivalent of short-crested waves in shallow water – and much more general patterns – have been studied as solutions of the Kadomtsev–Petviashvili (KP) equation, which is a model equation for weakly oblique shallow water waves. A comprehensive study of three-dimensional shallow-water waves in the context of the KP model has been given by Bryant (1982), Segur & Finkel (1985), Hammack, Scheffner & Segur (1989), Hammack *et al.* (1995) and Dubrovin, Flickinger & Segur (1997). A review of three-dimensional long waves is given in Akylas (1994).

The starting point for the analysis in this paper is the inviscid, irrotational, constant-density equations for steady water waves governed by a velocity potential and free-surface elevation. However, we formulate these equations in a new way. In § 2, we show that the complete set of equations for steady water waves can be expressed in the compact form

$$\mathbf{J}(\boldsymbol{u})Z_x + \mathbf{K}(\boldsymbol{v})Z_y = \nabla S(\boldsymbol{Z}), \tag{1.1}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the fluid velocities in the x and y horizontal directions respectively at the free surface, $S(\boldsymbol{Z})$ is a functional, and $\boldsymbol{Z}$ is a vector of dependent variables (including the surface elevation and the velocity potential, but other variables as well). A central observation in the construction is that $\mathbf{J}(\boldsymbol{u})$ and $\mathbf{K}(\boldsymbol{v})$ are skew-symmetric operators – with respect to a particular inner product – and, more importantly, the products $\mathbf{J}(\boldsymbol{u})\partial_x$ and $\mathbf{K}(\boldsymbol{v})\partial_y$ have the remarkable property that they are *symmetric operators*! The form (1.1) is a generalized Hamiltonian formulation known as a *Hamiltonian system on a multi-symplectic structure*.

One of the main reasons for identifying a variational or symplectic structure is that they lead to operators that are symmetric. Recall the situation in finite dimensions. A standard Hamiltonian system on $\mathbb{R}^{2n}$ can be written in the form

$$\mathbf{J}\, Z_t = \nabla H(\boldsymbol{Z}), \quad \boldsymbol{Z} \in \mathbb{R}^{2n}, \tag{1.2}$$

where $\nabla H(\boldsymbol{Z})$ is the gradient of the function H with respect to a standard inner product on $\mathbb{R}^{2n}$, and $\mathbf{J}$ is the unit skew-symmetric matrix:

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{1.3}$$

The system (1.2) is 'symmetric' in the sense that $\mathbf{J}\mathrm{d}/\mathrm{d}t$ is a symmetric operator – even though $\mathbf{J}$ is skew-symmetric – and the Hessian of H is a symmetric matrix when evaluated on any solution $\widehat{\boldsymbol{Z}}$. Symplecticity – as represented by the skew-symmetric matrix $\mathbf{J}$ – assures immediately that (1.2) is the Euler–Lagrange equation associated with a functional. To see this, let $\boldsymbol{Z} = (q, p)$, with $q, p \in \mathbb{R}^n$ and define the one-form ω and its parametrization by $\Theta(\boldsymbol{Z}, \boldsymbol{Z}_t)$:

$$\omega = \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} \quad \text{and} \quad \Theta(\boldsymbol{Z}, \boldsymbol{Z}_t) = \boldsymbol{p} \cdot \boldsymbol{q}_t. \tag{1.4}$$

Then using the standard inner product on $\mathbb{R}^{2n}$ – denoted by $\langle \cdot, \cdot \rangle$ – it is straightforward to verify that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Theta(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z},\boldsymbol{Z}_t+\varepsilon\,\mathrm{d}\boldsymbol{Z}_t)|_{\varepsilon=0}=\langle\mathbf{J}\boldsymbol{Z}_t,\,\mathrm{d}\boldsymbol{Z}\rangle+\frac{\mathrm{d}}{\mathrm{d}t}\omega$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}H(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z})\Big|_{\varepsilon=0}=\langle\nabla H(\boldsymbol{Z}),\,\mathrm{d}\boldsymbol{Z}\rangle.$$

Hence, taking variations $\mathrm{d}\boldsymbol{Z}(t)$ which vanish at the endpoints t_1, t_2, (1.2) follows from the first variation of the functional

$$\mathscr{F}(\boldsymbol{Z})=\int_{t_1}^{t_2}[\Theta(\boldsymbol{Z},\boldsymbol{Z}_t)-H(\boldsymbol{Z})]\,\mathrm{d}t, \tag{1.5}$$

that is, $(\mathrm{d}/\mathrm{d}\varepsilon)\mathscr{F}(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z})|_{\varepsilon=0}=0$ implies (1.2). Variational principles have been widely used in the theory of water waves, starting with the Lagrangian formulation of Luke (1967) and the averaged Lagrangian theory of Whitham (cf. Whitham 1974). A variational formulation of the problem of short-crested waves has been provided by Marchant & Roberts (1988, 1990) to predict the dynamics of short-crested waves. They formulated the problem of the nonlinear interaction between incident and reflected wavetrains by using the slowly varying averaged Lagrangian approach described in the book by Whitham (1974). They considered two cases: the case where the wavefield varies in the radial direction but has no angular variation, and the case where the wavefield varies in the angular direction but has no radial variation. Mollo-Christensen (1981) also applied Whitham's theory to study the stability of weakly nonlinear short-crested waves. However, as pointed out by Roberts (1983), his analysis is deficient because it ignores modulations with a component in the other horizontal direction and uses Whitham's theory which is only appropriate for a wavefield defined by a single phase function, whereas short-crested waves need two phase functions.

The Hamiltonian approach, based on the Zakharov (1968) formulation, has also been widely used in the study of water waves, and has been applied to short-crested waves by Badulin *et al.* (1995). In atmospheric and geophysical fluid dynamics, Hamiltonian methods are widely used (cf. Salmon 1998).

The multi-symplectic formulation of water waves is a generalization of the Zakharov (1968) symplectic structure. Indeed restriction of the full multi-symplectic structure to the time direction alone recovers the Zakharov symplectic structure (cf. Bridges 1996, p. 543). In the multi-symplectic setting, a distinct symplectic operator is assigned for *each space direction* as well as time (cf. Bridges 1996, 1997*a*, *b*, 1998), and this provides much more structure and information about the equations. For the steady patterns to be considered in this paper, the generalization of the above argument in (1.2)–(1.5) leads to a PDE with a skew-symmetric operator associated with each space direction, as shown in (1.1). This form of the equations leads to several new results. In § 3 we associate symmetry with uniform flow – where uniform flow corresponds to a constant depth h_0 and a constant vector-valued velocity (u_0, v_0) – and by associating mean flow with uniform flow, a new variational principle for uniform flows is derived which relies crucially on the form (1.1).

By coupling the argument in § 3 with a variational characterization of doubly periodic patterns, in § 4, we present a new formulation for steady doubly periodic patterns on a finite-depth fluid with a novel characterization of the coupled mean flow. In fact, this uniform pattern is characterized by a new constrained variational principle, where the constraints are the mass-flux vector and the Bernoulli functional.

An important consequence of the variational principle is a precise organization of the parameter structure: the theory shows that there are precisely ten parameters which come in dual pairs of which five are fixed (one of each dual pair). As far as we are aware this is the first time that the parameter structure for steady three-dimensional doubly periodic waves has been precisely identified. The wavenumbers of the pattern are Lagrange multipliers associated with the constraints of constant wave-action flux. This formulation is applied in § 6 to give a new characterization of short-crested Stokes waves interacting with a mean flow. In this context an *interaction between a periodic wave and a mean flow* means that the two classes of solutions – (i) a mean depth and mean velocity field and (ii) the periodic wave – exist as a coupled system. No time-dependent interaction is implied here – only dependence between the two solutions.

The formation of a doubly periodic wave on the surface of the ocean is closer in spirit to the subject of pattern formation than to wave mechanics. Therefore we use the terms wave and pattern interchangeably, with preference for pattern when discussing multi-periodic states.

A doubly periodic pattern can be expressed as a double Fourier series, and – unlike a Fourier series in one variable – such solutions are associated with a doubly periodic lattice (i.e. rhombic, rectangular, hexagonal). There is intriguing evidence to suggest that hexagonal waves as well as rhombic (or diamond) patterns are important in the theory of shallow water oceanographic patterns. For example Hammack *et al.* (1989, 1995) present a compelling argument in favour of hexagonal patterns, and Allen (1984, p. 398) presents fascinating evidence for rhombic patterns. Both of these patterns deserve further study as solutions of the full water wave problem – particularly their stability properties. In this paper we concentrate on rhombic patterns which are a model for short-crested waves.

In § 6, we restrict the variational principle of § 4 to a rhombic periodic lattice. At the linear level this leads to a pattern height of the form

$$\eta(x, y) = A_1 e^{i(\kappa x + \ell y)} + A_2 e^{i(\kappa x - \ell y)} + \overline{A_1} e^{-i(\kappa x + \ell y)} + \overline{A_2} e^{-i(\kappa x - \ell y)}, \tag{1.6}$$

where A_1 and A_2 are complex amplitudes. A weakly nonlinear analysis starting with this form shows that there are two classes of nonlinear patterns: oblique travelling waves ($A_2 = 0$ and $A_1 \neq 0$ or $A_1 = 0$ and $A_2 \neq 0$) and short-crested waves ($|A_1| = |A_2|$). In other words both classes of waves are included in the same analysis. One consequence of this approach is that it is possible to find secondary bifurcations that connect these two families at finite amplitude. It is important to note that the limiting process in which both oblique waves become parallel is tricky and does not provide pure travelling waves – but this limit is not considered in this paper (see Roberts 1983 and Roberts & Peregrine 1983 for an analysis of this limit). However we do consider another interesting limit: oblique travelling waves are obtained when one wave component, A_1 or A_2, vanishes.

An overview of the paper is as follows. In § 2 the governing equations for steady three-dimensional inviscid irrotational water waves with gravity and surface tension are recalled. These equations are then formulated as a Hamiltonian system on a multi-symplectic structure. In § 3 we introduce the concept of a uniform pattern and a variational principle for such patterns. In § 4 a global (for arbitrary amplitude) variational principle for doubly periodic patterns coupled with a mean flow is obtained. Particular cases of such patterns are short-crested waves in finite depth. In § 5 we analyse the dispersion relation for the linearization about the flat state, and in §§ 6 and 7 we apply the variational principle of § 4 to weakly nonlinear patterns which extend the form (1.6) to nonlinear patterns. We do not consider the linear stability of patterns, but we discuss in § 8 how the present formulation can contribute to a linear stability analysis.

2. Governing equations for steady three-dimensional capillary–gravity patterns

The starting point for the analysis is the governing equations for steady three-dimensional (two surface directions in addition to the vertical direction) patterns at the surface of a constant-density inviscid irrotational fluid, including surface-tension effects at the surface.

Let $(x, y) \in \mathbb{R}^2$ denote the horizontal coordinates and $z \in \mathbb{R}$, with $0 \leqslant z \leqslant \eta(x, y)$, denote the vertical coordinate, where $\eta(x, y)$ is the single-valued position of the surface. With velocity potential $\phi(x, y, z)$, the governing equations and boundary conditions for steady patterns take the following well-known form:

$$\Delta\phi \stackrel{\text{def}}{=} \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for} \quad 0 < z < \eta(x, y). \tag{2.1}$$

At the bottom $z = 0$, the boundary condition is

$$\phi_z = 0 \quad \text{at} \quad z = 0. \tag{2.2}$$

The kinematic condition at the free surface is

$$\phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{at} \quad z = \eta(x, y) \tag{2.3}$$

and the dynamic condition at the free surface is

$$\tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \sigma\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\right) - p = 0 \quad \text{at} \quad z = \eta(x, y) \tag{2.4}$$

where

$$w_1 \stackrel{\text{def}}{=} \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \quad \text{and} \quad w_2 \stackrel{\text{def}}{=} \frac{\eta_y}{\sqrt{1 + \eta_x^2 + \eta_y^2}}, \tag{2.5}$$

and the function p satisfies

$$p_x = 0 \quad \text{and} \quad p_y = 0. \tag{2.6}$$

While the equations (2.6) suggest that p is a constant, it is important to distinguish between the function p and its value. The value of p is the Bernoulli constant. The parameters g and σ represent the gravitational and surface-tension coefficients respectively.

In the remainder of this section, the governing equations are reformulated in order to derive a natural structure for the analysis of patterns. This reformulation involves introducing new variables. The new variables make it possible to cast the equations into a standard form from which some new interesting results are deduced. However it is important to keep in mind that, once the results are deduced, they can be transformed back into the standard coordinates. Let

$$u = \phi_x, \quad v = \phi_y, \quad \boldsymbol{u} = u|_{z=\eta}, \quad \boldsymbol{v} = v|_{z=\eta}, \quad \Phi = \phi|_{z=\eta}, \tag{2.7}$$

and

$$\eta = \nabla \cdot \gamma = \frac{\partial\gamma_1}{\partial x} + \frac{\partial\gamma_2}{\partial y}. \tag{2.8}$$

Then collect the set of dependent variables into the vector-valued function $\boldsymbol{Z}$:

$$(\Phi, \eta, \gamma_1, \gamma_2, p, w_1, w_2, \phi, u, v). \tag{2.9}$$

Note that p is included in (2.9) as a dependent variable. However, the most curious new variables are γ_1 and γ_2. The vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ is a vector-valued potential for the free surface, reminiscent of the potential for the velocity field. While a potential for the velocity field is widely used, the idea of using a potential-type field for the free surface elevation is new. This form for the surface elevation has interesting theoretical implications: it leads to a geometrical characterization of the mean flow.

We show that the governing equations can be written in the form

$$\mathbf{J}(\boldsymbol{u})Z_x + \mathbf{K}(\boldsymbol{v})Z_y = \nabla S(\boldsymbol{Z}), \tag{2.10}$$

where the vector-valued function $\boldsymbol{Z}$ is defined in (2.9), $\mathbf{J}(\boldsymbol{u})$ and $\mathbf{K}(\boldsymbol{v})$ are skew-symmetric operators, and $S(\boldsymbol{Z})$ is a scalar-valued function. Moreover the form of the equations (2.10) follows from the first variation of the functional

$$\mathscr{F}(\boldsymbol{Z}) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} [\Theta_1(\boldsymbol{Z}, Z_x) + \Theta_2(\boldsymbol{Z}, Z_y) - S(\boldsymbol{Z})]\,\mathrm{d}x\,\mathrm{d}y; \tag{2.11}$$

that is, $(\mathrm{d}/\mathrm{d}\varepsilon)\mathscr{F}(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z})|_{\varepsilon=0} = 0$ leads to (2.10). Explicit expressions for $\mathbf{J}(\boldsymbol{u})$, $\mathbf{K}(\boldsymbol{v})$ and $\nabla S(\boldsymbol{Z})$ are given in Appendix A.

To verify this new form of the governing equations and to define the terms involved, we need an inner product: for ten-component vector-valued functions of the form (2.9), let

$$\langle \boldsymbol{Z}, \boldsymbol{W}\rangle = \sum_{j=1}^{7} Z_j W_j + \int_0^{\eta} (Z_8 W_8 + Z_9 W_9 + Z_{10} W_{10})\,\mathrm{d}z. \tag{2.12}$$

Introduce the following pair of one-forms:

$$\omega_1 = \int_0^{\eta} u\,\mathrm{d}\phi\,\mathrm{d}z + \sigma w_1\,\mathrm{d}\eta + p\,\mathrm{d}\gamma_1 \quad \text{and} \quad \omega_2 = \int_0^{\eta} v\,\mathrm{d}\phi\,\mathrm{d}z + \sigma w_2\,\mathrm{d}\eta + p\,\mathrm{d}\gamma_2, \tag{2.13}$$

and the functional

$$S(\boldsymbol{Z}) = \int_0^{\eta} \tfrac{1}{2}(u^2 + v^2 - \phi_z^2)\,\mathrm{d}z - \tfrac{1}{2}g\eta^2 + p\eta + \sigma\left(1 - \sqrt{1 - w_1^2 - w_2^2}\right). \tag{2.14}$$

In the integrand of the functional $\mathscr{F}(Z)$ in (2.11), the terms $\Theta_1(Z, Z_x)$ and $\Theta_2(Z, Z_y)$ are parametrizations of the one-forms ω_1 and ω_2:

$$\left.\begin{aligned} \Theta_1(\boldsymbol{Z}, Z_x) &= \int_0^{\eta} u\phi_x\,\mathrm{d}z + \sigma w_1 \frac{\partial \eta}{\partial x} + p\frac{\partial \gamma_1}{\partial x}, \\ \Theta_2(\boldsymbol{Z}, Z_y) &= \int_0^{\eta} v\phi_y\,\mathrm{d}z + \sigma w_2 \frac{\partial \eta}{\partial y} + p\frac{\partial \gamma_2}{\partial y}. \end{aligned}\right\} \tag{2.15}$$

Now, using the inner product (2.12) it is straightforward to verify that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Theta_1(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z}, Z_x + \varepsilon\,\mathrm{d}Z_x)\big|_{\varepsilon=0} = \langle \mathbf{J}(\boldsymbol{u})Z_x,\, \mathrm{d}\boldsymbol{Z}\rangle + \frac{\partial}{\partial x}\omega_1,$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Theta_2(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z}, Z_y + \varepsilon\,\mathrm{d}Z_y)\big|_{\varepsilon=0} = \langle \mathbf{K}(\boldsymbol{v})Z_y,\, \mathrm{d}\boldsymbol{Z}\rangle + \frac{\partial}{\partial y}\omega_2,$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}S(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z})\big|_{\varepsilon=0} = \langle \nabla S(\boldsymbol{Z}),\, \mathrm{d}\boldsymbol{Z}\rangle,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathscr{F}(\boldsymbol{Z}+\varepsilon\,\mathrm{d}\boldsymbol{Z})\big|_{\varepsilon=0} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \langle \mathbf{J}(\boldsymbol{u})Z_x + \mathbf{K}(\boldsymbol{v})Z_y - \nabla S(\boldsymbol{Z}),\, \mathrm{d}\boldsymbol{Z}\rangle\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_{y_1}^{y_2}\int_{x_1}^{x_2} \left(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

By taking variations $\mathrm{d}Z$ which vanish at the endpoints x_1, x_2, y_1, y_2, the second term on the right-hand side vanishes, verifying that the first variation of $\mathscr{F}(Z)$ leads to (2.10).

It remains to verify that the system of ten equations (2.10) recovers the steady equations (2.1)–(2.6). Writing out the system of ten equations in (2.10) leads to

$$-\boldsymbol{u}\eta_x - \boldsymbol{v}\eta_y = -\phi_z|_{z=\eta}, \tag{2.16}$$

$$\boldsymbol{u}\Phi_x + \boldsymbol{v}\Phi_y - \sigma(w_1)_x - \sigma(w_2)_y = \tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \phi_z^2)|_{z=\eta} - g\eta + p, \tag{2.17}$$

$$-p_x = 0, \quad -p_y = 0, \tag{2.18}$$

$$(\gamma_1)_x + (\gamma_2)_y = \eta, \tag{2.19}$$

$$\sigma\eta_x = \sigma w_1/\sqrt{1 - w_1^2 - w_2^2}, \tag{2.20}$$

$$\sigma\eta_y = \sigma w_2/\sqrt{1 - w_1^2 - w_2^2}, \tag{2.21}$$

$$-u_x - v_y = \phi_{zz}, \tag{2.22}$$

$$\phi_x = u, \quad \phi_y = v. \tag{2.23}$$

Equation (2.16) is the kinematic condition (2.3) in terms of the new coordinates; equation (2.18) recovers (2.6); equation (2.19) recovers the definition (2.8); equations (2.20)–(2.21) recover the definitions (2.5); equation (2.22) is Laplace's equation (2.1) in terms of the new coordinates and (2.23) recovers the definitions in (2.7). To verify (2.17), note that

$$\Phi_x = [\phi_x + \phi_z\eta_x]|_{z=\eta} \quad \text{and} \quad \Phi_y = [\phi_y + \phi_z\eta_y]|_{z=\eta},$$

and so

$$\boldsymbol{u}\Phi_x + \boldsymbol{v}\Phi_y = (\phi_x^2 + \phi_y^2 + \phi_z^2)|_{z=\eta}.$$

Using this identity, the dynamic condition (2.4) is recast as

$$\boldsymbol{u}\Phi_x + \boldsymbol{v}\Phi_y - \tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \phi_z^2)|_{z=\eta} + g\eta - \sigma[(w_1)_x + (w_2)_y] - p = 0,$$

which, upon rearrangement, is (2.17). Accompanying the system (2.10) are the boundary conditions in (2.7) and the bottom boundary condition (2.2).

Equation (2.10) and the basic functionals (2.14)–(2.15) are the starting point for the analysis in the paper. The system (2.10) is a Hamiltonian system on a multi-symplectic structure (cf. Bridges 1996, 1997*a*, *b*, 1998). The operators $\mathbf{J}(\boldsymbol{u})$ and $\mathbf{K}(\boldsymbol{v})$ are skew symmetric – with respect to the inner product (2.12) – and $\nabla S(\boldsymbol{Z})$ is the gradient of the 'Hamiltonian' functional $S(\boldsymbol{Z})$. Multi-symplecticity is a generalization of Hamiltonian structure which is natural for the analysis of pattern formation equations, because it distinguishes between symplecticity in the x-direction and the y-direction: neglecting Z_y reduces (2.10) to a standard Hamiltonian system in the x-direction and neglecting Z_x reduces (2.10) to a standard Hamiltonian system in the y-direction.

A feature of the equations (2.10) that is important in what follows for characterizing the mean flow is the presence of symmetry. The symmetries we use are the classical ones (cf. Benjamin & Olver 1982). However, we decompose the conservation laws in a new way which contributes to a geometric characterization of mean flow. Our argument is that the mean flow corresponds to drift along the group orbit of a symmetry of the equations.

It is apparent from (2.6) that the function p is a conserved quantity. Moreover, the steady equations should conserve mass flux. In the multi-symplectic setting, these conservation laws are related to symmetry in the following way. Let

$$\left.\begin{aligned}\xi_1 &= (1,0,0,0,0,0,0,1,0,0)^T,\\ \xi_2 &= (0,0,1,0,0,0,0,0,0,0)^T,\\ \xi_3 &= (0,0,0,1,0,0,0,0,0,0)^T.\end{aligned}\right\} \tag{2.24}$$

Then it is straightforward to verify that

$$\mathbf{J}(\boldsymbol{u})\xi_1 = \nabla Q_1(\boldsymbol{Z}), \quad \mathbf{J}(\boldsymbol{u})\xi_2 = \nabla R(\boldsymbol{Z}), \quad \mathbf{J}(\boldsymbol{u})\xi_3 = 0, \tag{2.25}$$

$$\mathbf{K}(\boldsymbol{v})\xi_1 = \nabla Q_2(\boldsymbol{Z}), \quad \mathbf{K}(\boldsymbol{v})\xi_2 = 0, \quad \mathbf{K}(\boldsymbol{v})\xi_3 = \nabla R(\boldsymbol{Z}), \tag{2.26}$$

where

$$R(\boldsymbol{Z}) = p, \quad Q_1(\boldsymbol{Z}) = \int_0^\eta u\,\mathrm{d}z \quad \text{and} \quad Q_2(\boldsymbol{Z}) = \int_0^\eta v\,\mathrm{d}z. \tag{2.27}$$

The functional $R(\boldsymbol{Z})$ is the Bernoulli functional and its value is the Bernoulli constant, and the functionals $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$ are the components of the mass flux. The equations (2.25)–(2.26) connect these functionals with symmetry. The vectors ξ_1, ξ_2 and ξ_3 in (2.24) are generators for a three-parameter affine group with action

$$G_\theta \boldsymbol{Z} = \boldsymbol{Z} + \theta_1\xi_1 + \theta_2\xi_2 + \theta_3\xi_3 \quad \forall\ \theta = (\theta_1,\theta_2,\theta_3) \in \mathbb{R}^3. \tag{2.28}$$

The functional $S(\boldsymbol{Z})$ and the operators $\mathbf{J}(\boldsymbol{u})$ and $\mathbf{K}(\boldsymbol{v})$, and hence equation (2.10), are invariant with respect to the action of this group. For example

$$S(G_\theta \boldsymbol{Z}) = S\left(\boldsymbol{Z} + \sum_{j=1}^{3}\theta_j\xi_j\right) = S(\boldsymbol{Z}), \quad \forall\ \theta \in \mathbb{R}^3. \tag{2.29}$$

The fact that (2.25)–(2.26) assure conservation of mass flux and conservation of the Bernoulli functional follows by differentiating (2.29) with respect to each θ_j and using the identities (2.25); for example with $j = 1$,

$$\begin{aligned} 0 = \frac{\partial}{\partial\theta_1} S\left(\boldsymbol{Z} + \sum_{j=1}^{3}\theta_j\xi_j\right)\Bigg|_{\theta=0} &= \langle \nabla S(\boldsymbol{Z}), \xi_1\rangle \\ &= \langle \mathbf{J}(\boldsymbol{u})Z_x + \mathbf{K}(\boldsymbol{v})Z_y, \xi_1\rangle \\ &= -\langle Z_x, \mathbf{J}(\boldsymbol{u})\xi_1\rangle - \langle Z_y, \mathbf{K}(\boldsymbol{v})\xi_1\rangle \\ &= -\langle Z_x, \nabla Q_1(\boldsymbol{Z})\rangle - \langle Z_y, \nabla Q_2(\boldsymbol{Z})\rangle \\ &= -\frac{\partial Q_1}{\partial x} - \frac{\partial Q_2}{\partial y}, \end{aligned}$$

which corresponds to mass flux conservation. A similar argument with $j = 2, 3$ results in $\partial_x R(\boldsymbol{Z}) = 0$ and $\partial_y R(\boldsymbol{Z}) = 0$. In the next two sections, these three symmetries are related first to uniform patterns in § 3 and then to the interaction problem between steady doubly periodic patterns and a mean flow.

Another conservation law, which is not used in this paper but follows in a straightforward way from this new formulation of the steady equations, is the *conservation of wave-action flux*. The concept of wave action and wave-action flux was introduced by Whitham (cf. Whitham 1974, Chap. 16), and it has since found wide application in oceanographic and atmospheric flows (cf. Grimshaw 1984 and references therein), and a geometric formulation was given in Bridges (1997*b*). A geometric formulation of conservation of wave-action flux (since only the steady equations are considered here) in the present setting is obtained as follows.

Let $\boldsymbol{Z}(x, y, z; s)$ be a one-parameter family of solutions of (2.10) parametrized by s and 2π-periodic in s,

$$\boldsymbol{Z}(x, y, z; s+2\pi) = \boldsymbol{Z}(x, y, z; s).$$

Then, conservation of wave-action flux takes the form

$$\frac{\partial}{\partial x}\oint \Theta_1(\boldsymbol{Z}, Z_s)\,\mathrm{d}s + \frac{\partial}{\partial y}\oint \Theta_2(\boldsymbol{Z}, Z_s)\,\mathrm{d}s = 0, \tag{2.30}$$

where $\oint \mathrm{d}s$ is an integral over one period in s, and Θ_1 and Θ_2 are the functions introduced in (2.11). The conservation law (2.30) is easily verified by using the definition of $\Theta_1(\cdot,\cdot)$ and $\Theta_2(\cdot,\cdot)$ and noting that

$$\frac{\partial}{\partial s}S(\boldsymbol{Z}) = \langle \nabla S(\boldsymbol{Z}), Z_s\rangle = \langle \mathbf{J}(\boldsymbol{u})Z_x, Z_s\rangle + \langle \mathbf{K}(\boldsymbol{v})Z_y, Z_s\rangle,$$

and

$$\frac{\partial}{\partial x}\Theta_1(\boldsymbol{Z}, Z_s) = \frac{\partial}{\partial s}\Theta_1(\boldsymbol{Z}, Z_x) - \langle \mathbf{J}(\boldsymbol{u})Z_x, Z_s\rangle,$$

$$\frac{\partial}{\partial y}\Theta_2(\boldsymbol{Z}, Z_s) = \frac{\partial}{\partial s}\Theta_2(\boldsymbol{Z}, Z_y) - \langle \mathbf{K}(\boldsymbol{v})Z_y, Z_s\rangle.$$

Hence

$$\frac{\partial}{\partial x}\Theta_1(\boldsymbol{Z}, Z_s) + \frac{\partial}{\partial y}\Theta_2(\boldsymbol{Z}, Z_s) = \frac{\partial}{\partial s}[\Theta_1(\boldsymbol{Z}, Z_x) + \Theta_2(\boldsymbol{Z}, Z_y) - S(\boldsymbol{Z})],$$

and the right-hand side vanishes when integrated over a period in s. Because of the importance of wave-action conservation for periodic waves, there is good reason to conjecture that the conservation of wave-action flux is important for the analysis of steady doubly periodic patterns, but this is not considered here.

3. Uniform patterns

The purpose of this section is twofold: to show in the simplest possible setting how symmetry is crucially related to mean flow, and secondly to generalize the concept of *uniform flows* in hydraulics to two horizontal space dimensions – *uniform patterns*. Uniform flows are defined to be solutions of the steady equations (2.10) which are independent of x and y.

Setting $Z_x = Z_y = 0$ in (2.10) shows that such states correspond to critical points of the functional $S(\boldsymbol{Z})$. Using the expression for $\nabla S(\boldsymbol{Z})$ in Appendix A, it is clear that the only solution of $\nabla S(\boldsymbol{Z}) = 0$ is the trivial state,

$$u = v = \eta = p = 0 \quad \text{with} \quad \phi,\ \Phi,\ \gamma_1 \text{ and } \gamma_2 \text{ constant.}$$

However there is a second class of uniform flows which are associated with the three-parameter symmetry group (2.27). Let

$$\boldsymbol{Z}(x, y) = G_{\theta(x,y)}\boldsymbol{Z}_0 = \boldsymbol{Z}_0 + \theta_1(x, y)\xi_1 + \theta_2(x, y)\xi_2 + \theta_3(x, y)\xi_3, \tag{3.1}$$

where $\boldsymbol{Z}_0$ is some constant vector to be determined. This expression corresponds to an (x, y)-dependent flow along the G_θ group orbit. Take this flow to be linear in x

and y. We are interested in characterizing the mean velocity and elevation; therefore let

$$\theta_1(x,y) = u_0 x + v_0 y + \theta_1^o, \quad \theta_2(x,y) = h_0 x + \theta_2^o \quad \text{and} \quad \theta_3 = \theta_3^o, \tag{3.2}$$

where θ_1^o, θ_2^o and θ_3^o are arbitrary real numbers. Then

$$\mathbf{J}(\boldsymbol{u})Z_x = \mathbf{J}(\boldsymbol{u})\left[\frac{\partial\theta_1}{\partial x}\boldsymbol{\xi}_1 + \frac{\partial\theta_2}{\partial x}\boldsymbol{\xi}_2\right] = u_0\mathbf{J}(\boldsymbol{u})\boldsymbol{\xi}_1 + h_0\mathbf{J}(\boldsymbol{u})\boldsymbol{\xi}_2 = u_0\nabla Q_1(\boldsymbol{Z}_0) + h_0\nabla R(\boldsymbol{Z}_0),$$

using (2.25); similarly,

$$\mathbf{K}(\boldsymbol{v})Z_y = \mathbf{K}(\boldsymbol{v})\left[\frac{\partial\theta_1}{\partial y}\boldsymbol{\xi}_1\right] = v_0\mathbf{K}(\boldsymbol{v})\boldsymbol{\xi}_1 = v_0\nabla Q_2(\boldsymbol{Z}_0).$$

The function $S(\boldsymbol{Z})$ is G_θ-invariant and so $\nabla S(G_{\theta(x,y)}\boldsymbol{Z}_0) = \nabla S(\boldsymbol{Z}_0)$. Therefore substitution of (3.1) and (3.2) into (2.10) results in the following equation for $\boldsymbol{Z}_0$:

$$\nabla S(\boldsymbol{Z}_0) = u_0\nabla Q_1(\boldsymbol{Z}_0) + v_0\nabla Q_2(\boldsymbol{Z}_0) + h_0\nabla R(\boldsymbol{Z}_0). \tag{3.3}$$

This equation could have been obtained by simply substituting $u = u_0$, $v = v_0$ and $h = h_0$ into the governing steady equations. However this would not produce the form of the equations (3.3). The form (3.3) is interesting because it has the form of the Lagrange necessary condition for a constrained variational principle, with (h_0, u_0, v_0) as Lagrange multipliers. Note that θ_3^o has no dynamic significance and can therefore be set to zero.

We show that (h_0, u_0, v_0) correspond to a uniform pattern; that is, a constant vector-velocity field (u_0, v_0) and a constant depth h_0. In the present context the values of the functionals $R(\boldsymbol{Z}_0)$, $Q_1(\boldsymbol{Z}_0)$ and $Q_2(\boldsymbol{Z}_0)$ determine values of the mean quantities h_0, u_0 and v_0. In other words: a uniform pattern of the above form corresponds to a critical point of $S(\boldsymbol{Z}_0)$ restricted to level sets of the three functionals $R(\boldsymbol{Z}_0)$, $Q_1(\boldsymbol{Z}_0)$ and $Q_2(\boldsymbol{Z}_0)$. This state is uniform because the x and y dependence in (3.1) is along the group orbit: the potentials ϕ, γ_1 and γ_2 are linear functions of x and y.

Using the expressions for ∇S, ∇R, ∇Q_1 and ∇Q_2, it is straightforward to solve (3.3) for $\boldsymbol{Z}_0$; we find

$$\boldsymbol{Z}_0 = (0, h_0, 0, 0, p_0, 0, 0, 0, u_0, v_0)^T \quad \text{with} \quad p_0 = gh_0 + \tfrac{1}{2}(u_0^2 + v_0^2), \tag{3.4}$$

and h_0, u_0 and v_0 are determined by the values of the constraint sets:

$$r = R(\boldsymbol{Z}_0) = p_0, \quad q_1 = Q_1(\boldsymbol{Z}_0) = u_0 h_0 \quad \text{and} \quad q_2 = Q_2(\boldsymbol{Z}_0) = v_0 h_0, \tag{3.5}$$

where (r, q_1, q_2) correspond to values of the three functionals R, Q_1 and Q_2. Evaluating the functional S on this state we find

$$S_0 = S(\boldsymbol{Z}_0) = p_0 h_0 + \tfrac{1}{2}h_0(u_0^2 + v_0^2) - \tfrac{1}{2}gh_0^2 = \tfrac{1}{2}gh_0^2 + h_0(u_0^2 + v_0^2).$$

The above constrained variational principle is non-degenerate precisely when the Jacobian of (R, Q_1, Q_2) with respect to the parameters (h_0, u_0, v_0) is non-zero. This Jacobian takes the form

$$\begin{bmatrix} \dfrac{\partial R}{\partial h_0} & \dfrac{\partial R}{\partial u_0} & \dfrac{\partial R}{\partial v_0} \\ \dfrac{\partial Q_1}{\partial h_0} & \dfrac{\partial Q_1}{\partial u_0} & \dfrac{\partial Q_1}{\partial v_0} \\ \dfrac{\partial Q_2}{\partial h_0} & \dfrac{\partial Q_2}{\partial u_0} & \dfrac{\partial Q_2}{\partial v_0} \end{bmatrix} = \begin{bmatrix} g & u_0 & v_0 \\ u_0 & h_0 & 0 \\ v_0 & 0 & h_0 \end{bmatrix}, \tag{3.6}$$

and

$$\det\begin{bmatrix} \dfrac{\partial R}{\partial h_0} & \dfrac{\partial R}{\partial u_0} & \dfrac{\partial R}{\partial v_0} \\ \dfrac{\partial Q_1}{\partial h_0} & \dfrac{\partial Q_1}{\partial u_0} & \dfrac{\partial Q_1}{\partial v_0} \\ \dfrac{\partial Q_2}{\partial h_0} & \dfrac{\partial Q_2}{\partial u_0} & \dfrac{\partial Q_2}{\partial v_0} \end{bmatrix} = gh_0^2[1-(u_0^2+v_0^2)gh_0]. \tag{3.7}$$

If we define a Froude number based on the magnitude of the velocity, $[(u_0^2+v_0^2)/gh_0]^{1/2}$, then this determinant vanishes precisely when the Froude number is unity. In other words, degeneracy of the variational principle for uniform patterns is related to a form of criticality of the uniform patterns. A natural generalization of sub-criticality – from uniform flows in one space dimension to uniform patterns in two space dimensions – is

$$\det\begin{bmatrix} \dfrac{\partial R}{\partial h_0} & \dfrac{\partial R}{\partial u_0} & \dfrac{\partial R}{\partial v_0} \\ \dfrac{\partial Q_1}{\partial h_0} & \dfrac{\partial Q_1}{\partial u_0} & \dfrac{\partial Q_1}{\partial v_0} \\ \dfrac{\partial Q_2}{\partial h_0} & \dfrac{\partial Q_2}{\partial u_0} & \dfrac{\partial Q_2}{\partial v_0} \end{bmatrix} > 0. \tag{3.8}$$

Although subcriticality is important for determining when periodic gravity waves can occur in the linearization about a uniform flow in one space dimension, we show in § 5 that criticality plays a less important role in the linearization about uniform patterns.

The Lagrange multiplier theory gives further parameter structure. In particular,

$$h_0 = \frac{\partial S_0}{\partial r}, \quad u_0 = \frac{\partial S_0}{\partial q_1} \quad \text{and} \quad v_0 = \frac{\partial S_0}{\partial q_2}.$$

Therefore an equivalent condition for subcriticality is

$$\det[\mathrm{Hess}(S_0)] = \det\begin{bmatrix} \dfrac{\partial^2 S_0}{\partial r^2} & \dfrac{\partial^2 S_0}{\partial r \partial q_1} & \dfrac{\partial^2 S_0}{\partial r \partial q_2} \\ \dfrac{\partial^2 S_0}{\partial q_1 \partial r} & \dfrac{\partial^2 S_0}{\partial q_1^2} & \dfrac{\partial^2 S_0}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 S_0}{\partial q_2 \partial r} & \dfrac{\partial^2 S_0}{\partial q_2 \partial q_1} & \dfrac{\partial^2 S_0}{\partial q_2^2} \end{bmatrix} > 0. \tag{3.9}$$

To summarize, there is a three-parameter family of uniform patterns of the form

$$\boldsymbol{Z}(x,y) = \boldsymbol{Z}_0 + \theta_1(x,y)\boldsymbol{\xi}_1 + \theta_2(x)\boldsymbol{\xi}_2, \tag{3.10}$$

with $\theta_1(x,y)$ and $\theta_2(x)$ given by (3.2) and Z_0 given by (3.4) as a function of (h_0,u_0,v_0) with the values of the mean quantities determined by the values of the parameters (r,q_1,q_2). Note that θ_3^o has been set to zero, since it has no dynamic significance. This uniform pattern is non-degenerate precisely when the determinant (3.7) is non-zero, and this condition is related to criticality for uniform patterns in two space dimensions. In the next section, we consider the coupled problem, where the uniform pattern is changed by a nonlinear doubly periodic pattern.

4. Steady doubly periodic patterns coupled to a mean flow

In the case of one horizontal space dimension, periodic travelling waves can be viewed in two equivalent ways. If the basic fluid is considered to be at rest, the travelling wave is steady relative to a reference frame moving at the wave speed. On the other hand, if the basic flow is a uniform flow of depth h_0 and speed u_0, the wave can be considered as a stationary periodic wave on the uniform flow, in a fixed reference frame. It is this latter view that we generalize to two horizontal space dimensions in this section.

When a doubly periodic pattern forms on a uniform pattern, the two patterns are coupled, leading to changes in the uniform pattern – that is, a pattern-generated mean flow. In this section, the coupled problem is formulated. The complete pattern is composed of a doubly periodic pattern with wavenumber (κ, ℓ), a mean elevation h_0 and a vector-valued velocity (u_0, v_0).

In general, doubly periodic patterns are supported by a periodic lattice which is constructed with the symmetry of any of the basic wallpaper groups; in particular the basic lattice is either rectangular, rhombic or hexagonal (cf. Armstrong 1988, § 25). If the basic fluid state is taken to be quiescent, there is no preferred direction in the horizontal plane and therefore any of these lattices would be admissible. However, for doubly periodic patterns on a uniform pattern, there is a preferred direction imposed by the uniform velocity field (u_0, v_0). The preferred direction restricts the lattice type to be rhombic. In fact such patterns correspond precisely to short-crested Stokes waves in finite depth interacting with a mean flow; that is, steady rhombic patterns superposed on (and interacting with) a uniform pattern correspond to a new characterization of short-crested Stokes waves in finite depth. Moreover, this characterization is for short-crested waves of arbitrary amplitude.

The basic pattern (3.1) is generalized as follows. Let

$$\boldsymbol{Z}(x,y) = G_{\theta(x,y)}\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y}) = \widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y}) + \theta_1(x,y)\boldsymbol{\xi}_1 + \theta_2(x)\boldsymbol{\xi}_2, \tag{4.1}$$

where $\theta_1(x,y)$ and $\theta_2(x)$ are defined exactly as in (3.2), although the values of h_0, u_0 and v_0 are determined by the coupled problem here. The function $\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y})$ is 2π-periodic in $\widehat{x}$ and $\widehat{y}$ and

$$\widehat{x} = \kappa x + \widehat{x}^o \quad \text{and} \quad \widehat{y} = \ell y + \widehat{y}^o, \tag{4.2}$$

where $\widehat{x}^o$ and $\widehat{y}^o$ are arbitrary real numbers and (κ, ℓ) is the wavenumber vector for the steady doubly periodic pattern. To determine an equation for $\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y})$ we substitute (4.1) into (2.10). Considering each term in (2.10) separately we find

$$\begin{aligned}
\mathbf{J}(\boldsymbol{u})Z_x &= \kappa\mathbf{J}(\boldsymbol{u})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{x}} + u_0\mathbf{J}(\boldsymbol{u})\boldsymbol{\xi}_1 + h_0\mathbf{J}(\boldsymbol{u})\boldsymbol{\xi}_2 \\
&= \kappa\mathbf{J}(\boldsymbol{u})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{x}} + u_0\nabla Q_1(\widehat{\boldsymbol{Z}}) + h_0\nabla R(\widehat{\boldsymbol{Z}}), \\
\mathbf{K}(\boldsymbol{v})Z_y &= \ell\mathbf{K}(\boldsymbol{v})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{y}} + v_0\mathbf{K}(\boldsymbol{v})\boldsymbol{\xi}_1 = \ell\mathbf{K}(\boldsymbol{v})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{y}} + v_0\nabla Q_2(\widehat{\boldsymbol{Z}}),
\end{aligned}$$

and since $S(\boldsymbol{Z})$ is G_θ-invariant, $\nabla S(\boldsymbol{Z}) = \nabla S(\widehat{\boldsymbol{Z}})$. Therefore (2.10) reduces to

$$\kappa\mathbf{J}(\boldsymbol{u})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{x}} + \ell\mathbf{K}(\boldsymbol{v})\frac{\partial\widehat{\boldsymbol{Z}}}{\partial\widehat{y}} = \nabla S(\widehat{\boldsymbol{Z}}) - h_0\nabla R(\widehat{\boldsymbol{Z}}) - u_0\nabla Q_1(\widehat{\boldsymbol{Z}}) - v_0\nabla Q_2(\widehat{\boldsymbol{Z}}). \tag{4.3}$$

This equation is the governing equation for the doubly periodic function $\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y})$. However, equation (4.3) can also be viewed as the Lagrange necessary condition for a constrained variational principle.

We showed in § 2 that the gradient of the functional with integrand $\Theta_1(\boldsymbol{Z},Z_x)+\Theta_2(\boldsymbol{Z},Z_y)$ resulted in the left-hand side of (2.10). Therefore define

$$\mathscr{A}_1(\widehat{\boldsymbol{Z}})=\int_{\mathbb{T}^2}\Theta_1(\widehat{\boldsymbol{Z}},\widehat{Z}_{\widehat{x}})\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}\quad\text{and}\quad\mathscr{A}_2(\widehat{\boldsymbol{Z}})=\int_{\mathbb{T}^2}\Theta_2(\widehat{\boldsymbol{Z}},\widehat{Z}_{\widehat{y}})\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y},\tag{4.4}$$

where

$$\int_{\mathbb{T}^2}\cdot\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}=1(2\pi)^2\int_0^{2\pi}\int_0^{2\pi}\cdot\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}.\tag{4.5}$$

Since we are now interested in doubly periodic functions, we include integration over $\mathbb{T}^2$ in the inner product:

$$[\boldsymbol{W},\boldsymbol{Z}]\stackrel{\text{def}}{=}\int_{\mathbb{T}^2}\langle\boldsymbol{W},\boldsymbol{Z}\rangle\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}\tag{4.6}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in (2.12).

The functionals $\mathscr{A}_1(\widehat{\boldsymbol{Z}})$ and $\mathscr{A}_2(\widehat{\boldsymbol{Z}})$ correspond to the two components of wave-action flux evaluated on a doubly periodic pattern.

With respect to the inner product (4.6), the left-hand side of (4.3) can be characterized as the gradient of a functional; in particular, (4.3) takes the form

$$\nabla\mathscr{S}(\widehat{\boldsymbol{Z}})=\kappa\nabla\mathscr{A}_1(\widehat{\boldsymbol{Z}})+\ell\nabla\mathscr{A}_2(\widehat{\boldsymbol{Z}})+h_0\nabla\mathscr{R}(\widehat{\boldsymbol{Z}})+u_0\nabla\mathscr{Q}_1(\widehat{\boldsymbol{Z}})+v_0\nabla\mathscr{Q}_2(\widehat{\boldsymbol{Z}}),\tag{4.7}$$

where the gradients of $\mathscr{R}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are to be interpreted with respect to the inner product (4.6). Script symbols are used for functionals that are averaged over $\mathbb{T}^2$.

Doubly-periodic patterns coupled to a mean flow can therefore be characterized as critical points of $\mathscr{S}$ restricted to level sets of the five functionals: $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{R}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$, with κ, ℓ, h_0, u_0 and v_0 as Lagrange multipliers, and (4.7) is the Lagrange necessary condition. The non-degeneracy condition for this variational principle is a generalization of that in § 3:

$$\det\begin{bmatrix}\dfrac{\partial\mathscr{A}_1}{\partial\kappa}&\dfrac{\partial\mathscr{A}_1}{\partial\ell}&\dfrac{\partial\mathscr{A}_1}{\partial h_0}&\dfrac{\partial\mathscr{A}_1}{\partial u_0}&\dfrac{\partial\mathscr{A}_1}{\partial v_0}\\[2mm]\dfrac{\partial\mathscr{A}_2}{\partial\kappa}&\dfrac{\partial\mathscr{A}_2}{\partial\ell}&\dfrac{\partial\mathscr{A}_2}{\partial h_0}&\dfrac{\partial\mathscr{A}_2}{\partial u_0}&\dfrac{\partial\mathscr{A}_2}{\partial v_0}\\[2mm]\dfrac{\partial\mathscr{R}}{\partial\kappa}&\dfrac{\partial\mathscr{R}}{\partial\ell}&\dfrac{\partial\mathscr{R}}{\partial h_0}&\dfrac{\partial\mathscr{R}}{\partial u_0}&\dfrac{\partial\mathscr{R}}{\partial v_0}\\[2mm]\dfrac{\partial\mathscr{Q}_1}{\partial\kappa}&\dfrac{\partial\mathscr{Q}_1}{\partial\ell}&\dfrac{\partial\mathscr{Q}_1}{\partial h_0}&\dfrac{\partial\mathscr{Q}_1}{\partial u_0}&\dfrac{\partial\mathscr{Q}_1}{\partial v_0}\\[2mm]\dfrac{\partial\mathscr{Q}_2}{\partial\kappa}&\dfrac{\partial\mathscr{Q}_2}{\partial\ell}&\dfrac{\partial\mathscr{Q}_2}{\partial h_0}&\dfrac{\partial\mathscr{Q}_2}{\partial u_0}&\dfrac{\partial\mathscr{Q}_2}{\partial v_0}\end{bmatrix}\neq 0.\tag{4.8}$$

The Lagrange multiplier theory gives further parameter structure; in particular

$$\kappa=\frac{\partial\mathscr{S}}{\partial a_1},\quad\ell=\frac{\partial\mathscr{S}}{\partial a_2},\quad h_0=\frac{\partial\mathscr{S}}{\partial r},\quad u_0=\frac{\partial\mathscr{S}}{\partial q_1}\quad\text{and}\quad v_0=\frac{\partial\mathscr{S}}{\partial q_2},\tag{4.9}$$

where a_1, a_2, r, q_1 and q_2 are the values of the constraint sets $\mathscr{A}_1(\widehat{\boldsymbol{Z}})$, $\mathscr{A}_2(\widehat{\boldsymbol{Z}})$, $\mathscr{R}(\widehat{\boldsymbol{Z}})$,

$\mathscr{Q}_1(\widehat{\boldsymbol{Z}})$ and $\mathscr{Q}_2(\widehat{\boldsymbol{Z}})$. Hence, an equivalent non-degeneracy condition is

$$\det[\mathrm{Hess}(\mathscr{S})] = \det \begin{bmatrix} \dfrac{\partial^2 \mathscr{S}}{\partial a_1^2} & \dfrac{\partial^2 \mathscr{S}}{\partial a_1 \partial a_2} & \dfrac{\partial^2 \mathscr{S}}{\partial a_1 \partial r} & \dfrac{\partial^2 \mathscr{S}}{\partial a_1 \partial q_1} & \dfrac{\partial^2 \mathscr{S}}{\partial a_1 \partial q_2} \\ \dfrac{\partial^2 \mathscr{S}}{\partial a_2 \partial a_1} & \dfrac{\partial^2 \mathscr{S}}{\partial a_2^2} & \dfrac{\partial^2 \mathscr{S}}{\partial a_2 \partial r} & \dfrac{\partial^2 \mathscr{S}}{\partial a_2 \partial q_1} & \dfrac{\partial^2 \mathscr{S}}{\partial a_2 \partial q_2} \\ \dfrac{\partial^2 \mathscr{S}}{\partial r \partial a_1} & \dfrac{\partial^2 \mathscr{S}}{\partial r \partial a_2} & \dfrac{\partial^2 \mathscr{S}}{\partial r^2} & \dfrac{\partial^2 \mathscr{S}}{\partial r \partial q_1} & \dfrac{\partial^2 \mathscr{S}}{\partial r \partial q_2} \\ \dfrac{\partial^2 \mathscr{S}}{\partial q_1 \partial a_1} & \dfrac{\partial^2 \mathscr{S}}{\partial q_1 \partial a_2} & \dfrac{\partial^2 \mathscr{S}}{\partial q_1 \partial r} & \dfrac{\partial^2 \mathscr{S}}{\partial q_1^2} & \dfrac{\partial^2 \mathscr{S}}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 \mathscr{S}}{\partial q_2 \partial a_1} & \dfrac{\partial^2 \mathscr{S}}{\partial q_2 \partial a_2} & \dfrac{\partial^2 \mathscr{S}}{\partial q_2 \partial r} & \dfrac{\partial^2 \mathscr{S}}{\partial q_2 \partial q_1} & \dfrac{\partial^2 \mathscr{S}}{\partial q_2^2} \end{bmatrix} \neq 0. \quad (4.10)$$

There are other ways to interpret the equation (4.7). For example, the values of κ, ℓ, h_0, u_0 and v_0 can be prescribed (instead of the values of the functionals). In this case, the functional

$$\mathscr{F}(\widehat{\boldsymbol{Z}}, \kappa, \ell, h_0, u_0, v_0) = \mathscr{S}(\widehat{\boldsymbol{Z}}) - \kappa \mathscr{A}_1(\widehat{\boldsymbol{Z}}) - \ell \mathscr{A}_2(\widehat{\boldsymbol{Z}}) - h_0 \mathscr{R}(\widehat{\boldsymbol{Z}}) - u_0 \mathscr{Q}_1(\widehat{\boldsymbol{Z}}) - v_0 \mathscr{Q}_2(\widehat{\boldsymbol{Z}})$$

is an unconstrained functional, and the values of the functionals $\mathscr{A}_1(\widehat{\boldsymbol{Z}})$, $\mathscr{A}_2(\widehat{\boldsymbol{Z}})$, $\mathscr{R}(\widehat{\boldsymbol{Z}})$, $\mathscr{Q}_1(\widehat{\boldsymbol{Z}})$ and $\mathscr{Q}_2(\widehat{\boldsymbol{Z}})$ would be determined by the solution. In general, the parameters come in pairs:

$$(\kappa, \mathscr{A}_1) \quad (\ell, \mathscr{A}_2) \quad (h_0, \mathscr{R}) \quad (u_0, \mathscr{Q}_1) \quad (v_0, \mathscr{Q}_2), \quad (4.11)$$

and one of each pair can be prescribed – but not both – and the other of each pair is determined by the solution. In other words, the variational structure organizes the ten parameters – which define the pattern–mean flow interaction – in a precise way.

To see how the matrices in (4.8) and (4.10) contribute to the analysis, consider the pattern–mean flow interaction problem near the flat state. At the flat state, the functionals $\mathscr{A}_1$ and $\mathscr{A}_2$ vanish and the functionals $\mathscr{R}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ take values, r, q_1 and q_2 respectively, associated with the uniform pattern (h_0, u_0, v_0). Now, evaluate each functional $\mathscr{A}_1, \ldots, \mathscr{Q}_2$ on this solution and expand these functionals in a Taylor series near the flat state:

$$\begin{pmatrix} \mathscr{A}_1(\kappa, \ell, h_0, u_0, v_0) \\ \mathscr{A}_2(\kappa, \ell, h_0, u_0, v_0) \\ \mathscr{R}(\kappa, \ell, h_0, u_0, v_0) \\ \mathscr{Q}_1(\kappa, \ell, h_0, u_0, v_0) \\ \mathscr{Q}_2(\kappa, \ell, h_0, u_0, v_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \\ q_1 \\ q_2 \end{pmatrix} + \begin{bmatrix} \dfrac{\partial \mathscr{A}_1}{\partial \kappa} & \dfrac{\partial \mathscr{A}_1}{\partial \ell} & \dfrac{\partial \mathscr{A}_1}{\partial h_0} & \dfrac{\partial \mathscr{A}_1}{\partial u_0} & \dfrac{\partial \mathscr{A}_1}{\partial v_0} \\ \dfrac{\partial \mathscr{A}_2}{\partial \kappa} & \dfrac{\partial \mathscr{A}_2}{\partial \ell} & \dfrac{\partial \mathscr{A}_2}{\partial h_0} & \dfrac{\partial \mathscr{A}_2}{\partial u_0} & \dfrac{\partial \mathscr{A}_2}{\partial v_0} \\ \dfrac{\partial \mathscr{R}}{\partial \kappa} & \dfrac{\partial \mathscr{R}}{\partial \ell} & \dfrac{\partial \mathscr{R}}{\partial h_0} & \dfrac{\partial \mathscr{R}}{\partial u_0} & \dfrac{\partial \mathscr{R}}{\partial v_0} \\ \dfrac{\partial \mathscr{Q}_1}{\partial \kappa} & \dfrac{\partial \mathscr{Q}_1}{\partial \ell} & \dfrac{\partial \mathscr{Q}_1}{\partial h_0} & \dfrac{\partial \mathscr{Q}_1}{\partial u_0} & \dfrac{\partial \mathscr{Q}_1}{\partial v_0} \\ \dfrac{\partial \mathscr{Q}_2}{\partial \kappa} & \dfrac{\partial \mathscr{Q}_2}{\partial \ell} & \dfrac{\partial \mathscr{Q}_2}{\partial h_0} & \dfrac{\partial \mathscr{Q}_2}{\partial u_0} & \dfrac{\partial \mathscr{Q}_2}{\partial v_0} \end{bmatrix} \begin{pmatrix} \delta \kappa \\ \delta \ell \\ \delta h_0 \\ \delta u_0 \\ \delta v_0 \end{pmatrix} + \cdots .$$

Then, if $(I_1, \ldots, I_5)$ are the prescribed values of the five functionals, the change in the wavenumbers and the mean field due to the pattern, to leading order, is

$$\begin{pmatrix} \delta\kappa \\ \delta\ell \\ \delta h_0 \\ \delta u_0 \\ \delta v_0 \end{pmatrix} = \begin{bmatrix} \frac{\partial\mathscr{A}_1}{\partial\kappa} & \frac{\partial\mathscr{A}_1}{\partial\ell} & \frac{\partial\mathscr{A}_1}{\partial h_0} & \frac{\partial\mathscr{A}_1}{\partial u_0} & \frac{\partial\mathscr{A}_1}{\partial v_0} \\ \frac{\partial\mathscr{A}_2}{\partial\kappa} & \frac{\partial\mathscr{A}_2}{\partial\ell} & \frac{\partial\mathscr{A}_2}{\partial h_0} & \frac{\partial\mathscr{A}_2}{\partial u_0} & \frac{\partial\mathscr{A}_2}{\partial v_0} \\ \frac{\partial\mathscr{R}}{\partial\kappa} & \frac{\partial\mathscr{R}}{\partial\ell} & \frac{\partial\mathscr{R}}{\partial h_0} & \frac{\partial\mathscr{R}}{\partial u_0} & \frac{\partial\mathscr{R}}{\partial v_0} \\ \frac{\partial\mathscr{Q}_1}{\partial\kappa} & \frac{\partial\mathscr{Q}_1}{\partial\ell} & \frac{\partial\mathscr{Q}_1}{\partial h_0} & \frac{\partial\mathscr{Q}_1}{\partial u_0} & \frac{\partial\mathscr{Q}_1}{\partial v_0} \\ \frac{\partial\mathscr{Q}_2}{\partial\kappa} & \frac{\partial\mathscr{Q}_2}{\partial\ell} & \frac{\partial\mathscr{Q}_2}{\partial h_0} & \frac{\partial\mathscr{Q}_2}{\partial u_0} & \frac{\partial\mathscr{Q}_2}{\partial v_0} \end{bmatrix}^{-1} \begin{pmatrix} I_1 \\ I_2 \\ I_3 - r \\ I_4 - q_1 \\ I_5 - q_2 \end{pmatrix} + \cdots \tag{4.12}$$

where $|I_1|$, $|I_2|$, $|I_3-r|$, $|I_4-q_1|$ and $|I_5-q_2|$ are small. The expression (4.12) determines the change in wavenumbers (deviating from the values of κ, ℓ associated with the dispersion relation; cf. § 5) and the deviation in the mean field associated with the doubly periodic pattern.

5. Linearization about uniform patterns: the dispersion relation

The dispersion relation for the linearization about a uniform pattern is given by

$$D(\kappa,\ell) \equiv (\kappa u_0 + \ell v_0)^2 - (g + \sigma K^2)K \tanh K h_0 = 0, \tag{5.1}$$

where κ and ℓ are the wavenumbers in the x- and y-directions respectively and

$$K = \sqrt{\kappa^2 + \ell^2}. \tag{5.2}$$

In the setting of §§ 3–4, this dispersion relation is derived as follows. Let (h_0, u_0, v_0, Z_0) be a fixed uniform pattern. Linearization of (4.3) about this pattern reduces to

$$\kappa\, \mathbf{J}(u_0)\widehat{Z}_{\hat{x}} + \ell\, \mathbf{K}(v_0)\widehat{Z}_{\hat{y}} = \mathbf{A}_0\, \widehat{\boldsymbol{Z}} \tag{5.3}$$

where $\mathbf{A}_0$ is the symmetric operator

$$\mathbf{A}_0 = \text{Hess}(S - h_0 R - u_0 Q_1 - v_0 Q_2)(\boldsymbol{Z}_0).$$

Let $\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y},z) = \text{Re}\,(\zeta(z)e^{i(\hat{x}+\hat{y})})$; then substitution into (5.3) leads to the multi-parameter eigenvalue problem (where κ and ℓ are both considered as eigenparameters)

$$\mathbf{A}_0\boldsymbol{\zeta} = i\kappa\, \mathbf{J}(u_0)\boldsymbol{\zeta} + i\ell\, \mathbf{K}(v_0)\boldsymbol{\zeta}. \tag{5.4}$$

Many of the properties of this eigenvalue problem are deduced from the fact that $\mathbf{A}_0$ is a symmetric operator and $\mathbf{J}(u_0)$ and $\mathbf{K}(v_0)$ are skew-symmetric operators. These properties are especially useful when the basic pattern is non-uniform. However, since the linearization is about a uniform pattern, this system is easily solved explicitly:

$$\boldsymbol{\zeta}(z) = (\widetilde{\Phi}, \widetilde{\eta}, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{p}, \widetilde{w}_1, \widetilde{w}_2, \widetilde{\phi}(z), \widetilde{u}(z), \widetilde{v}(z)),$$

with

$$\widetilde{\gamma}_1 = -i\widetilde{\eta}/\kappa, \quad \widetilde{\gamma}_2 = 0, \quad \widetilde{p} = 0, \quad \widetilde{w}_1 = i\kappa\widetilde{\eta}, \quad \widetilde{w}_2 = i\ell\widetilde{\eta},$$

$$\widetilde{\phi}(z) = \widetilde{\Phi}\frac{\cosh Kz}{\cosh Kh_0}, \quad \widetilde{u}(z) = i\kappa\widetilde{\phi}(z) \quad \text{and} \quad \widetilde{v}(z) = i\ell\widetilde{\phi}(z),$$

where $\widetilde{\Phi}$ and $\widetilde{\eta}$ satisfy

$$\begin{bmatrix} K \tanh Kh_0 & -\mathrm{i}(\kappa u_0 + \ell v_0) \\ \mathrm{i}(\kappa u_0 + \ell v_0) & g + \sigma K^2 \end{bmatrix} \begin{pmatrix} \widetilde{\Phi} \\ \widetilde{\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.5}$$

The dispersion relation is then obtained by setting the determinant of the Hermitian matrix in (5.5) to zero.

The dispersion relation is recast into the form

$$(u_0^2 - gh_0)\kappa^2 + 2u_0v_0\kappa\ell + (v_0^2 - gh_0)\ell^2 + gh_0K^2\left(1 - (1 + WK^2h_0^2)\frac{\tanh Kh_0}{Kh_0}\right) = 0,$$

where $W = \sigma(gh_0^2)^{-1}$ is the Weber number.

When $W = 0$ and $\ell = 0$ the dispersion relation reduces to

$$D(\kappa, 0) = (u_0^2 - gh_0)\kappa^2 + gh_0\kappa^2\left(1 - \frac{\tanh \kappa h_0}{\kappa h_0}\right) = 0,$$

with the property that there exist real non-zero wavenumbers only if the uniform flow is subcritical: $u_0^2 < gh_0$. Now, suppose $\ell \neq 0$ but $W = 0$ and try to generalize this result:

$$D(\kappa, \ell) = (\kappa \ell)\begin{bmatrix} u_0^2 - gh_0 & u_0v_0 \\ u_0v_0 & v_0^2 - gh_0 \end{bmatrix}\begin{pmatrix} \kappa \\ \ell \end{pmatrix} + gh_0K^2\left(1 - \frac{\tanh Kh_0}{Kh_0}\right) = 0. \tag{5.6}$$

The determinant of the matrix associated with the quadratic form in (κ, ℓ) is $-gh_0(u_0^2 + v_0^2 - gh_0)$ which is precisely the determinant in (3.7) (after multiplication by g). The natural definition of super-criticality would correspond to positivity of this matrix – but this matrix always has at least one negative eigenvalue ($-gh_0$) with the other equal to $u_0^2 + v_0^2 - gh_0$. In other words, criticality of uniform patterns does not play a clear role in determining when doubly periodic patterns can appear on a uniform pattern.

Given u_0, v_0, h_0, σ, and g, the fundamental requirement for the appearance of doubly periodic patterns is that $D(\kappa, \ell) = 0$ has real solutions. Real solutions of the dispersion relation (5.6) exist: this can be seen by diagonalizing the matrix associated with the quadratic form in (κ, ℓ) with a rotation matrix, and noting that this matrix always has at least one negative eigenvalue.

Another informative way that the dispersion relation can be recast is as

$$D(\kappa, \ell) = gh_0K^2\left[F^2\cos^2(\theta - \vartheta) - (1 + WK^2h_0^2)\frac{\tanh Kh_0}{Kh_0}\right] = 0,$$

where

$$u_0 = U_0 \cos\theta, \quad v_0 = U_0 \sin\theta, \quad \kappa = K\cos\vartheta \quad \text{and} \quad \ell = K\sin\vartheta,$$

and $F^2 = U_0^2/(gh_0)$. This form of the dispersion relation shows the importance of the angle between the vectors (u_0, v_0) and (κ, ℓ). When these vectors are aligned, the dispersion relation reduces to the one-dimensional case, and when they are orthogonal there are no real solutions of the dispersion relation. A complete analysis of this dispersion relation and its dependence on $\theta - \vartheta$ and W would be of great interest, but we do not consider this further here. Our main interest is in establishing that there exist values of (κ, ℓ) such that the dispersion relation (5.1) has solutions, in preparation for the study of nonlinear doubly periodic patterns near the flat state.

6. Capillary–gravity short-crested waves in finite depth

In this section, nonlinear doubly periodic patterns near the flat state, when the uniform velocity has a preferred direction, are studied using the theory of § 4 and we show that this leads to a new characterization of short-crested Stokes waves travelling on a finite-depth fluid, and includes oblique travelling waves in the same analysis.

The dispersion relation (5.1) is repeated here:

$$D(\kappa,\ell) = (\kappa u_0 + \ell v_0)^2 - (g + \sigma K^2) K \tanh K h_0. \tag{6.1}$$

For fixed u_0, v_0, h_0, g and σ, this dispersion relation is a function of κ^2 and ℓ^2. Therefore solutions come in pairs $(\pm\kappa, \pm\ell)$ and so for each (κ,ℓ) the height of the linearized doubly periodic pattern takes the general form

$$\eta(x,y) = A_1 \mathrm{e}^{\mathrm{i}(\kappa x+\ell y)} + A_2 \mathrm{e}^{\mathrm{i}(\kappa x-\ell y)} + \overline{A_1}\mathrm{e}^{-\mathrm{i}(\kappa x+\ell y)} + \overline{A_2}\mathrm{e}^{-\mathrm{i}(\kappa x-\ell y)}, \tag{6.2}$$

where A_1 and A_2 are complex amplitudes. In this section the weakly nonlinear patterns associated with this basic form are studied. This formulation contains both oblique Stokes capillary–gravity travelling waves (i.e. set $A_2 = 0$) as well as capillary–gravity short-crested Stokes waves, travelling in finite depth coupled with a mean flow.

For short-crested waves, when surface tension effects are neglected and the mean flow is restricted, the previous third-order results of Hsu *et al.* (1979) are recovered. In addition, the formulation leads to several new results on short-crested waves and the interaction between short-crested waves and oblique travelling waves. The effect of surface tension is new and the present formulation gives more comprehensive information about the parameter structure and the mean flow interaction. The form of the linear pattern (6.2) includes short-crested waves (when $|A_1| = |A_2| \neq 0$) as well as two families of oblique travelling waves (when $|A_1| \neq 0$ and $|A_2| = 0$ or $|A_1| = 0$ and $|A_2| \neq 0$). Consideration of both classes of waves leads to a coupled system of amplitude equations. In the analysis of these coupled amplitude equations, new degeneracies are found which show that near certain parameter values, oblique travelling waves and short-crested waves interact leading to a new class of waves bifurcating from short-crested waves.

To analyse the nonlinear problem, the vector-valued function $\widehat{Z}$ defined in (4.1) is expanded in a double Fourier series in $\hat{x}$ and $\hat{y}$, and the variational principle of § 4 is applied to solve for the coefficients in the Fourier series. In fact, the wave-height potentials and the velocity potential are expanded in Fourier series and the other seven components of $\widehat{Z}$ are determined from these potentials. Noting that $\widehat{\gamma}_1(\widehat{x},\widehat{y})$ and $\widehat{\gamma}_2(\widehat{x},\widehat{y})$ are 2π-periodic in $\widehat{x}$ and $\widehat{y}$, and have zero mean, the leading-order Fourier expansion for the wave-height potentials is taken to be of the form

$$\widehat{\gamma}_1(\widehat{x},\widehat{y}) = \mathrm{Re}\left[\frac{A_1}{\mathrm{i}\kappa}\mathrm{e}^{\mathrm{i}(\hat{x}+\hat{y})} + \frac{A_2}{\mathrm{i}\kappa}\mathrm{e}^{\mathrm{i}(\hat{x}-\hat{y})} + \frac{A_3}{2i\kappa}\mathrm{e}^{2\mathrm{i}(\hat{x}+\hat{y})} + \frac{A_4}{2i\kappa}\mathrm{e}^{2\mathrm{i}(\hat{x}-\hat{y})} + \frac{A_5}{2i\kappa}\mathrm{e}^{2\mathrm{i}\hat{x}}\right] + \cdots, \tag{6.3}$$

and

$$\widehat{\gamma}_2(\widehat{x},\widehat{y}) = \mathrm{Re}\left[\frac{A_6}{2\mathrm{i}\ell}\mathrm{e}^{2\mathrm{i}\hat{y}}\right] + \cdots, \tag{6.4}$$

where the Fourier coefficients $A_1, A_2, \ldots$ are complex amplitudes. There is considerable freedom in choosing the form for $\widehat{\gamma}_1(\widehat{x},\widehat{y})$ and $\widehat{\gamma}_2(\widehat{x},\widehat{y})$, but their precise form is not important as long as the function $\kappa\partial\widehat{\gamma}_1/\partial\widehat{x} + \ell\partial\widehat{\gamma}_2/\partial\widehat{y}$ gives the required form for the pattern height. With $\widehat{\gamma}_1(\widehat{x},\widehat{y})$ and $\widehat{\gamma}_2(\widehat{x},\widehat{y})$ of the form (6.3)–(6.4), the leading-order

Fourier expansion for the pattern height is

$$\widehat{\eta}(\widehat{x},\widehat{y}) = h_0 + \kappa\frac{\partial\widehat{\gamma}_1}{\partial\widehat{x}} + \ell\frac{\partial\widehat{\gamma}_2}{\partial\widehat{y}}$$

$$= h_0 + \mathrm{Re}\,[A_1\mathrm{e}^{\mathrm{i}(\hat{x}+\hat{y})} + A_2\mathrm{e}^{\mathrm{i}(\hat{x}-\hat{y})} + A_3\mathrm{e}^{2\mathrm{i}(\hat{x}+\hat{y})} + A_4\mathrm{e}^{2\mathrm{i}(\hat{x}-\hat{y})}$$

$$+A_5\mathrm{e}^{2\mathrm{i}\hat{x}} + A_6\mathrm{e}^{2\mathrm{i}\hat{y}}] + \cdots. \qquad (6.5)$$

For the velocity potential the leading-order Fourier expansion takes the form

$$\widehat{\phi}(\widehat{x},\widehat{y},z) = \mathrm{Re}\left[\frac{\cosh Kz}{\cosh Kh_0}(B_1\mathrm{e}^{\mathrm{i}(\hat{x}+\hat{y})} + B_2\mathrm{e}^{\mathrm{i}(\hat{x}-\hat{y})})\right.$$

$$+\frac{\cosh 2Kz}{\cosh 2Kh_0}(B_3\mathrm{e}^{2\mathrm{i}(\hat{x}+\hat{y})} + B_4\mathrm{e}^{2\mathrm{i}(\hat{x}-\hat{y})})$$

$$\left.+\frac{\cosh 2\kappa z}{\cosh 2\kappa h_0}B_5\mathrm{e}^{2\mathrm{i}\hat{x}} + \frac{\cosh 2\ell z}{\cosh 2\ell h_0}B_6\mathrm{e}^{2\mathrm{i}\hat{y}} + \cdots\right]. \qquad (6.6)$$

The variable p is independent of x and y and is therefore set at some constant value (here we set it equal to the value associated with the flat state). The other five components of the vector $\widehat{\boldsymbol{Z}}$ are determined from equation (4.1) to be

$$\widehat{\Phi} = \widehat{\phi}|_{z=\widehat{\eta}}, \quad \widehat{u} = u_0 + \kappa\widehat{\phi}_{\hat{x}}, \quad \widehat{v} = v_0 + \ell\widehat{\phi}_{\hat{y}},$$

and

$$\widehat{w}_1 = \kappa\widehat{\eta}_{\hat{x}}/\sqrt{1+\kappa^2\widehat{\eta}_{\hat{x}}^2+\ell^2\widehat{\eta}_{\hat{y}}^2}, \quad \widehat{w}_2 = \ell\widehat{\eta}_{\hat{y}}/\sqrt{1+\kappa^2\widehat{\eta}_{\hat{x}}^2+\ell^2\widehat{\eta}_{\hat{y}}^2},$$

with $\widehat{\phi}$, $\widehat{\gamma}_1$, $\widehat{\gamma}_2$ and $\widehat{\eta}$ given by the Fourier expansions (6.3)–(6.6).

The expressions (6.3)–(6.6) are substituted into the six functionals $\mathscr{S}$, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{R}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$. The explicit expressions for these functionals evaluated on the Fourier series are given in Appendix B. The Lagrange functional is then

$$\mathscr{F}(\widehat{\boldsymbol{Z}},\kappa,\ell,h_0,u_0,v_0) = \mathscr{S}(\widehat{\boldsymbol{Z}}) - \kappa\mathscr{A}_1(\widehat{\boldsymbol{Z}}) - \ell\mathscr{A}_2(\widehat{\boldsymbol{Z}}) - h_0\mathscr{R}(\widehat{\boldsymbol{Z}}) - u_0\mathscr{Q}_1(\widehat{\boldsymbol{Z}}) - v_0\mathscr{Q}_2(\widehat{\boldsymbol{Z}}),$$

where $\mathscr{F}$ is now considered as a function of the Fourier coefficients and Lagrange multipliers. Expressions for A_j, B_j for $j = 1, 2, \ldots$ are obtained by solving the algebraic equations

$$\frac{\partial\mathscr{F}}{\partial\overline{A_j}} = \frac{\partial\mathscr{F}}{\partial\overline{B_j}} = 0 \quad \text{for } j = 1, 2 \ldots.$$

The coefficients B_j ($j = 1, 2, \ldots$) are obtained by solving the equations $\partial\mathscr{F}/\partial\overline{B_j} = 0$,

$$B_1 = \mathrm{i}\frac{\kappa u_0 + \ell v_0}{KT_{10}}\left(A_1 - K\frac{T_{10}^2+2}{2T_{10}}\overline{A_1}A_3 - \frac{K^2}{8}\frac{T_{10}^2-2}{T_{10}^2}A_1|A_1|^2\right)$$

$$-\mathrm{i}\frac{(\kappa u_0 - \ell v_0)(\kappa^2-\ell^2)}{2K^2T_{10}^2}(\overline{A_2}A_5 + A_2A_6) - \mathrm{i}\frac{u_0\kappa^2}{KT_{10}T_{21}}\overline{A_2}A_5 - \mathrm{i}\frac{v_0\ell^2}{KT_{10}T_{22}}A_2A_6$$

$$+\mathrm{i}\frac{1}{4K^2T_{10}^2T_{21}T_{22}}\left[T_{22}(-u_0\kappa)(4\kappa^3 + KT_{10}T_{21}(11\kappa^2-\ell^2))\right.$$

$$\left.-v_0\ell T_{21}(KT_{10}T_{22}(-\kappa^2+3\ell^2) - 4\ell^3)\right]A_1|A_2|^2 + \cdots,$$

$$B_2 = \mathrm{i}\frac{\kappa u_0 - \ell v_0}{K T_{10}}\left(A_2 - K\frac{T_{10}^2+2}{2T_{10}}\overline{A_2}A_4 - \frac{K^2}{8}\frac{T_{10}^2-2}{T_{10}^2}A_2|A_2|^2\right)$$
$$-\mathrm{i}\frac{(\kappa u_0+\ell v_0)(\kappa^2-\ell^2)}{2K^2T_{10}^2}(\overline{A_1}A_5 + A_1\overline{A_6}) - \mathrm{i}\frac{\kappa u_0^2}{K T_{10}T_{21}}\overline{A_1}A_5 + \mathrm{i}\frac{v_0\ell^2}{K T_{10}T_{22}}A_1\overline{A_6}$$
$$+\mathrm{i}\frac{1}{4K^2T_{10}^2T_{21}T_{22}}\left[T_{22}(-u_0\kappa)(4\kappa^3 + K T_{10}T_{21}(11\kappa^2-\ell^2))\right.$$
$$\left.+v_0\ell T_{21}(K T_{10}T_{22}(-\kappa^2+3\ell^2) - 4\ell^3)\right] A_2|A_1|^2 + \cdots,$$

$$B_3 = \mathrm{i}\frac{\kappa u_0+\ell v_0}{K T_{20}}\left(A_3 - \frac{K}{2T_{10}}A_1^2\right)+\cdots,\quad B_4 = i\frac{\kappa u_0-\ell v_0}{K T_{20}}\left(A_4 - \frac{K}{2T_{10}}A_2^2\right)+\cdots,$$

$$B_5 = i\frac{u_0}{T_{21}}\left(A_5 - \frac{\kappa^2}{K T_{10}}A_1A_2\right)+\cdots,\quad B_6 = i\frac{v_0}{T_{22}}\left(A_6 - \frac{\ell^2}{K T_{10}}A_1\overline{A_2}\right)+\cdots.$$

The functional $\mathscr{F}$ is then written as a function of the coefficients A_j, $j = 1,\ldots,$ and expressions for these amplitudes are obtained by solving $\partial\mathscr{F}/\partial\overline{A_j} = 0$, for $j = 3,\ldots$; we find

$$A_3 = \frac{(\kappa u_0+\ell v_0)^2}{4T_{10}^2}\left(\frac{3-T_{10}^2}{gT_{10}^2 - \sigma K^2(3-T_{10}^2)}\right)A_1^2+\cdots,$$

$$A_4 = \frac{(\kappa u_0-\ell v_0)^2}{4T_{10}^2}\left(\frac{3-T_{10}^2}{gT_{10}^2 - \sigma K^2(3-T_{10}^2)}\right)A_2^2+\cdots,$$

$$A_5 = \frac{\kappa[(\kappa u_0)^2([3K^2T_{10}^2 - \kappa^2+\ell^2]T_{21} - 4K\kappa T_{10}) + \ell^2 v_0^2 T_{21}(K^2T_{10}^2+\kappa^2-\ell^2)]}{2K^2T_{10}^2(\kappa T_{21}[g+4\sigma\kappa^2] - 2K T_{10}[g+\sigma K^2])}A_1A_2$$
$$+\cdots,$$

$$A_6 = \frac{(\kappa u_0)^2(K^2T_{10}^2 - \kappa^2+\ell^2)T_{22} + (\ell v_0)^2([3K^2+\kappa^2-\ell^2]T_{22} - 4K\ell T_{10})}{2K^2T_{10}^2(T_{22}[g+4\sigma\ell^2] - 2\ell v_0^2)}A_1\overline{A_2}$$
$$+\cdots,$$

where

$$T_{10} = \tanh(Kh_0),\quad T_{20} = \tanh(2Kh_0),\quad T_{21} = \tanh(2\kappa h_0)\quad\text{and}\quad T_{22} = \tanh(2\ell h_0).$$

Back substitution of the Fourier coefficients into the Lagrange functional $\mathscr{F}$ results in

$$\mathscr{F} = \alpha_1|A_1|^2 + \alpha_1|A_2|^2 + \tfrac{1}{2}\alpha_2|A_1|^4 + \alpha_3|A_1|^2|A_2|^2 + \tfrac{1}{2}\alpha_2|A_2|^4 + \cdots. \tag{6.7}$$

Setting the derivatives $\partial\mathscr{F}/\partial\overline{A}_j = 0$ for $j = 1, 2$ yields the following system of amplitude equations:

$$\left.\begin{aligned}\alpha_1A_1 + \alpha_2A_1|A_1|^2 + \alpha_3A_1|A_2|^2 + \cdots = 0,\\ \alpha_1A_2 + \alpha_3A_2|A_1|^2 + \alpha_2A_2|A_2|^2 + \cdots = 0.\end{aligned}\right\} \tag{6.8}$$

These equations contain the leading-order bifurcation theory for the patterns with elevation (6.5) and velocity potential (6.6). To simplify the expressions, we rotate coordinates so that the uniform velocity (u_0, v_0) is aligned with the x-axis, i.e. $v_0 = 0$.

In this case, expressions for the coefficients in (6.8) are

$$\alpha_1 = \frac{\kappa^2 u_0^2 - gK\,T_{10}(1+\tau)}{K\,T_{10}},$$

$$\alpha_2 = gK^2\left(\tfrac{3}{8}\tau - \frac{(1+\tau)(2T_{10}^2-1)}{2T_{10}^2} - \frac{(1+\tau)^2(3-T_{10}^2)^2}{8T_{10}^2(T_{10}^2-\tau(3-T_{10}^2))} + \frac{(1+\tau)^2(1-T_{10}^2)^2}{4T_{10}^2}\right),$$

$$\alpha_3 = \tfrac{1}{4}\,gK^2\left(\tau\,(\,3\overline{\kappa}^4 + 3\overline{\ell}^4 - 2\overline{\kappa}^2\overline{\ell}^2\,) + \frac{(\,\overline{\ell}^2 - \overline{\kappa}^2 + T_{10}^2\,)^2\,(1+\tau)^2}{T_{10}^2\,(\,1+4\,\tau\,\overline{\ell}^2\,)} + \frac{(\,T_{10}^2-1)^2\,(1+\tau)^2}{T_{10}^2}\right.$$
$$\left. + \frac{\overline{\kappa}\,(1+\tau)^2\,[2]^2}{T_{10}^2\,T_{21}\,[1]} + 8\,\frac{\overline{\kappa}^3\,(1+\tau)}{T_{21}\,T_{10}} + 4\,(\overline{\ell}^2 - 3\overline{\kappa}^2)\,(1+\tau)\right),$$

where

$$\overline{\kappa} = \frac{\kappa}{K}, \quad \overline{\ell} = \frac{\ell}{K},$$
$$[1] = \overline{\kappa}\,T_{21}\,(\,1+4\,\tau\,\overline{\kappa}^2\,) - 2\,T_{10}(1+\tau),$$
$$[2] = (\,3\,T_{10}^2 + \overline{\ell}^2 - \overline{\kappa}^2)T_{21} - 4\,\overline{\kappa}\,T_{10},$$
$$\tau = \sigma K^2/g = W(Kh_0)^2,$$

and τ is the Bond number. See Menasce (1995) for complete details of these calculations, including the case $v_0 \neq 0$.

It is useful here to review the role of the parameters, with reference to the discussion surrounding equation (4.11). In the context of the constrained variational principle of §4, the equations (6.8) are an intermediate step, since it is the values of the constraint sets that are prescribed. This viewpoint is dramatically different from the classical view of amplitude equations. First, the coupled system (6.8) is solved for A_1 and A_2 as functions of the five parameters κ, ℓ, h_0, u_0 and v_0 (assuming that (6.8) is non-degenerate, an issue which we consider in detail below). All the Fourier coefficients are expressed as functions of these five parameters, resulting in the vector $\widehat{\boldsymbol{Z}}$ as a function of $(\widehat{x},\widehat{y})$ and these five parameters. Substitution into the constraint sets in (4.7) leads to the identities

$$\mathscr{A}_1(\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y};\kappa,\ell,h_0,u_0,v_0)) = a_1,$$
$$\mathscr{A}_2(\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y};\kappa,\ell,h_0,u_0,v_0)) = a_2,$$
$$\mathscr{R}(\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y};\kappa,\ell,h_0,u_0,v_0)) = r,$$
$$\mathscr{Q}_1(\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y};\kappa,\ell,h_0,u_0,v_0)) = q_1,$$
$$\mathscr{Q}_2(\widehat{\boldsymbol{Z}}(\widehat{x},\widehat{y};\kappa,\ell,h_0,u_0,v_0)) = q_2.$$

Inversion of these five equations, assuming the non-degeneracy condition (4.8) is satisfied, leads to κ, ℓ, h_0, u_0 and v_0 as functions of a_1, a_2, r, q_1 and q_2.

On the other hand, the values of the parameters κ, ℓ, h_0, u_0 and v_0 can be fixed (this is closer to the classical approach to the solution of amplitude equations). Which approach is taken depends on which is appropriate for the particular physical question: whether bulk quantities such as mass flux and pressure head are fixed or whether mean quantities such as elevation and mean velocity can be fixed. Here, in order to compare with the classical results of Hsu *et al.* (1979), we consider κ, ℓ, h_0, q_1 (or rather $|A_1|$) fixed, $v_0 = 0$, and determine u_0.

For $|A_1| + |A_2|$ sufficiently small, the solutions of (6.8) are classified as follows. The flat state corresponds to $|A_1| = |A_2| = 0$. Oblique travelling waves correspond to $|A_2| = 0$ and $|A_1| \neq 0$ (or $|A_1| \neq 0$ and $|A_2| = 0$) in which case (6.8) reduces to $\alpha_1 + \alpha_2|A_1|^2 + \cdots = 0$, or

$$u_0^2 = \frac{gK}{\kappa^2} T_{10}(1+\tau) - \frac{K T_{10}}{\kappa^2} \alpha_2 |A_1|^2 + \cdots,$$

which yields the well-known result for the change in the wave speed (identifying u_0 with the wave speed) of oblique capillary–gravity travelling waves.

When $|A_1| \neq 0$ and $|A_2| \neq 0$ the system (6.8), truncated at third order, reduces to

$$\begin{pmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix} \begin{pmatrix} |A_1|^2 \\ |A_2|^2 \end{pmatrix} = - \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix}. \tag{6.9}$$

This equation has a unique solution when $\alpha_2^2 \neq \alpha_3^2$ and the solution is

$$|A_1|^2 = |A_2|^2 = -\frac{\alpha_1}{\alpha_2 + \alpha_3}.$$

This class of solutions corresponds precisely to short-crested Stokes waves. Another way to write this expression is $\alpha_1 + (\alpha_2 + \alpha_3)|A_1|^2 = 0$ or

$$u_0^2 = \frac{gK}{\kappa^2} T_{10}(1+\tau) - \frac{K T_{10}}{\kappa^2} (\alpha_2 + \alpha_3)|A_1|^2 + \cdots, \tag{6.10}$$

where the $\cdots$ are a reminder that higher-order terms are being neglected. In the form (6.10), the amplitude correction of the 'wave-speed' u_0 is easily compared with the results of Hsu *et al.* (1979). With $\tau = 0$ and the restricted mean flow (i.e. $v_0 = 0$ and h_0 set at the flat state), it is straightforward to show that the expression for $\alpha_2 + \alpha_3$ reduces to the coefficient ω_2 of Hsu *et al.* (1979, equation (56)). Note also that the short-crested waves reduce to transverse standing waves when $\kappa = 0$, and in this case the present coefficient $\alpha_2 + \alpha_3$ reduces to equation (34) of Concus (1962) for capillary–gravity standing waves. Another check is provided when comparing the coefficient $\alpha_2 + \alpha_3$ with equation (5.26) of Hogan *et al.* (1988). We performed this comparison by taking the deep water limit of our results and the short-crested wave limit of the results of Hogan *et al.* (1988). Recall that they allow for different wavenumbers in the x- and y-directions.

When $\alpha_2 = \alpha_3$ or $\alpha_2 = -\alpha_3$ the determinant in (6.9) vanishes and higher-order terms need to be included. When $\alpha_2 + \alpha_3 < 0$, the phase velocity of the short-crested wave increases with amplitude, whereas when $\alpha_2 + \alpha_3 > 0$, it decreases. In water of infinite depth, in the absence of surface tension, $\alpha_2 + \alpha_3$ vanishes for $\overline{\kappa} = 0.374$. This result was discovered numerically by Roberts (1983) (see his figure 5). Here, this singularity is extended to the case of finite depth and non-zero surface tension and the results are shown in figure 2.

The regions of positive and negative values for $\alpha_2 + \alpha_3$ are shown in figure 2 in the plane with coordinates $(\tanh(m), \tanh(n))$ where $m = \kappa h_0$ and $n = \ell h_0$. The curves Γ_1 and Γ_3 identify a sign change in $\alpha_2 + \alpha_3$, hence for values of m and n lying on these curves $\alpha_2 + \alpha_3 = 0$. The curves Γ_2 and Γ_4 are the loci of the zeros of the denominator of $\alpha_2 + \alpha_3$. The consequence of this singularity is that there is more than one family of short-crested waves for a range of wave speeds near this singularity.

The curves Γ_2 and Γ_4 are associated with resonances of the linear problem: two

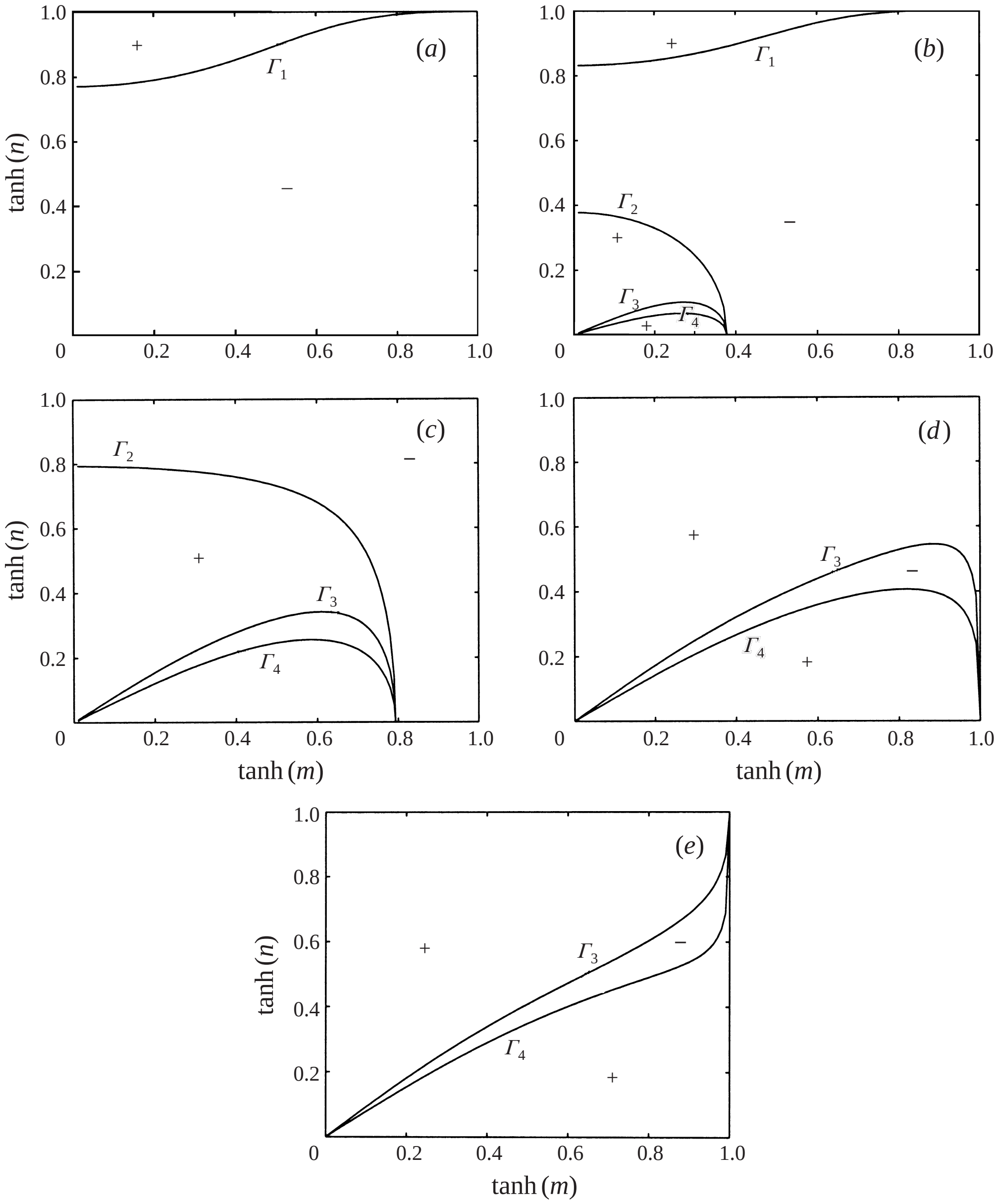

FIGURE 2. Sign of the coefficient $\alpha_2 + \alpha_3$ in (6.10) in the $[\tanh(\kappa h_0), \tanh(\ell h_0)]$-plane for a range of fixed values of the Bond number τ, which is proportional to surface tension. In each plot, the bottom-left corner corresponds to shallow water; the top-right to deep water; the top-left to near standing waves; and the bottom-right to nearly progressive waves. A positive value means that the phase velocity of the short-crested wave increases with amplitude. The curves Γ_1 and Γ_3 represent the zeros of $\alpha_2 + \alpha_3$ while Γ_2 and Γ_4 represent the poles of $\alpha_2 + \alpha_3$. These poles correspond to linear resonances. In water of infinite depth, in the absence of surface tension, $\alpha_2 + \alpha_3$ vanishes for $\overline{\kappa} = 0.374$. The values of the Bond number for each figure are (*a*) $\tau = 0$, (*b*) $\tau = 0.05$, (*c*) $\tau = \tau^*(= 0.2656)$, (*d*) $\tau = 0.5$, (*e*) $\tau = 0.7$.

waves of differing wavenumbers have the same linear frequency. For values of (m, n) on the curve Γ_2, defined by

$$\Gamma_2 = \{(m,n) : m^2 + n^2 = (\tanh^{-1})^2(\sqrt{3\tau/(1+\tau)})\},$$

the values $(2m, 2n)$ also satisfy the dispersion relation. This is the generalization of the well-known Wilton ripples to short-crested waves. For points close to the curve Γ_2, the values of A_3 and A_4 in the solution for the pattern height $\widehat{\eta}(\widehat{x},\widehat{y})$ (and B_3, B_4 in the expansion for $\widehat{\phi}(\widehat{x},\widehat{y},z)$) become very large. For values of (m, n) lying on the curve Γ_4, defined by

$$\Gamma_4 = \left\{(m,n) : \frac{(m^2+n^2+4\tau m^2)}{(m^2+n^2)^{3/2}} = \frac{(1+\tau)}{m}\frac{\tanh\sqrt{m^2+n^2}}{\tanh m}(1+\tanh^2 m)\right\},$$

the value $(2m, 0)$ also satisfies the dispersion relation, and for points near Γ_4 the values of A_5 and B_5 become very large.

Without surface tension the curves Γ_2, Γ_3 and Γ_4 are reduced to the origin, so the $(\tanh(m), \tanh(n))$-plane is divided in two regions by the curve Γ_1. When the Bond number τ is increased there are five regions as shown in figure 2. Note that for values of τ greater than $\tau^* \simeq 0.2656$ (τ^* is the real root of $136\tau^3 + 66\tau^2 + 3\tau - 8$) the curve Γ_1 becomes the whole line $\tanh(n) = 1$. For $\tau = 1/2$ the curve Γ_2 is the union of the two lines $\tanh(m) = 1$ and $\tanh(n) = 1$. There are only three domains left where $\alpha_2 + \alpha_3$ changes of sign. The regions of positive and negative values of $\alpha_2 + \alpha_3$ for values of τ greater than $1/2$ are shown on figure 2(*e*).

7. A new bifurcation of short-crested Stokes waves

The system (6.8) possesses solutions other than travelling waves or short-crested waves when $\alpha_2 = \alpha_3$, and these new solutions only become apparent by treating the oblique travelling waves and short-crested waves in the same analysis: indeed, it turns out that these solutions form a connecting branch between the short-crested waves and the oblique travelling waves.

When $\alpha_2 = \alpha_3$, the pair of amplitude equations (6.8) reduces to

$$\left.\begin{aligned} \alpha_1 + \alpha_2(|A_1|^2 + |A_2|^2) + \cdots = 0, \\ \alpha_1 + \alpha_2(|A_1|^2 + |A_2|^2) + \cdots = 0. \end{aligned}\right\} \quad (7.1)$$

For a non-degenerate solution, higher-order terms in the amplitude need to be computed. However, the general form of these coupled equations is known to arbitrary high order; that is, the equations are $\mathbb{D}_4$-equivariant. Therefore, even though we do not know the precise value of the coefficients of the higher-order terms, general conclusions can be drawn about behaviour of solutions. In fact this type of degeneracy has been studied by Bridges & Dias (1990) and Dias & Bridges (1994) in another context. The degeneracy $\alpha_2 = \alpha_3$ leads to a secondary bifurcation of solutions and, in the present context, this secondary branch connects the branch of short-crested waves with the branch of oblique travelling waves. Even in water of infinite depth, in the absence of surface tension, $\alpha_2 - \alpha_3$ can vanish: it vanishes for $\overline{\kappa} = 0.893$. As far as we are aware, this class of three-dimensional solutions has not been previously recognized.

In figure 3 one sees that, for a given Bond number not in the interval $[0.5, 0.649]$, there always exists a curve in parameter space where $\alpha_2 = \alpha_3$. While the degeneracy

$\alpha_2 = \alpha_3$ occurs only on a curve, the secondary bifurcation of new waves persists at finite amplitude for large regions of parameter space and should therefore be observable.

The branch of secondary waves is expected to also be multi-periodic in space but the amplitudes of the two basic modes, $|A_1|$ and $|A_2|$, depart from being equal as the branch bifurcates from the branch of short-crested waves. Therefore, near the parameter values associated with the singularity, the new waves appear as steady but slightly distorted short-crested waves, but at large amplitude they may appear quite different.

8. Concluding remarks

In this paper, a new formulation of the problem of steady doubly periodic patterns on a finite-depth fluid, including the effects of a uniform mean flow, is presented. One of the consequences of this formulation is a new characterization of short-crested waves and a framework within which short-crested waves and oblique travelling waves are analysed together. By analysing the coupled problem, including short-crested and oblique travelling waves, we found – already at low amplitude – that there is a region in parameter space with a secondary branch connecting these two families of waves.

One of the difficulties with analysing waves and patterns in a finite-depth fluid is the role of parameters. The variational formulation presented in § 4 showed that there are naturally ten parameters which come in dual pairs. Therefore a branch of such waves requires specification of exactly five parameters. Although the variational structure was important for identifying it, this parameter structure can be usefully applied to other methods for the analysis of steady patterns. For example, these properties of the parameter structure should also be useful for numerical computation of large-amplitude steady patterns in finite depth.

The theory in this paper can contribute to a linear stability theory for short-crested waves in shallow water – or in general, any steady multi-periodic pattern – in several ways. A fundamental question about the stability of waves and patterns in shallow water is the implication of mean flow for stability and instability. For example, in two space dimensions (one horizontal), it is known that the Stokes travelling wave is stable to the Benjamin–Feir mechanism when the depth is sufficiently small, and this stabilization is due to mean flow effects. But the Stokes wave is unstable even in shallow water when transverse perturbations are included. However no comprehensive study of the role of mean flow for large-amplitude waves or waves with a significant mean flow component has been given. For example, the seminal numerical study of McLean (1982) considered the mean flow frozen at a particular value and then varied only the wave parameters. Therefore it is an open question whether large variations in mean flow can alter dramatically the stability of even two-dimensional waves. In three space dimensions (two horizontal), experimental evidence of Hammack *et al.* (1989, 1995) strongly suggests that there are large classes of multi-periodic patterns – particularly hexagonal patterns – which are stable in shallow water. The formulation in this paper points out that the parameter space for steady multi-periodic patterns is quite large, and in this parameter space there may be several regions of stability and instability. A division of this parameter space into such regions would be of great interest.

The parameter structure of multi-periodic patterns can contribute to a linear stability analysis in another way. In Bridges (1996) it is shown that Jacobian matrices on parameter space such as (4.8) contain information about linear stability exponents,

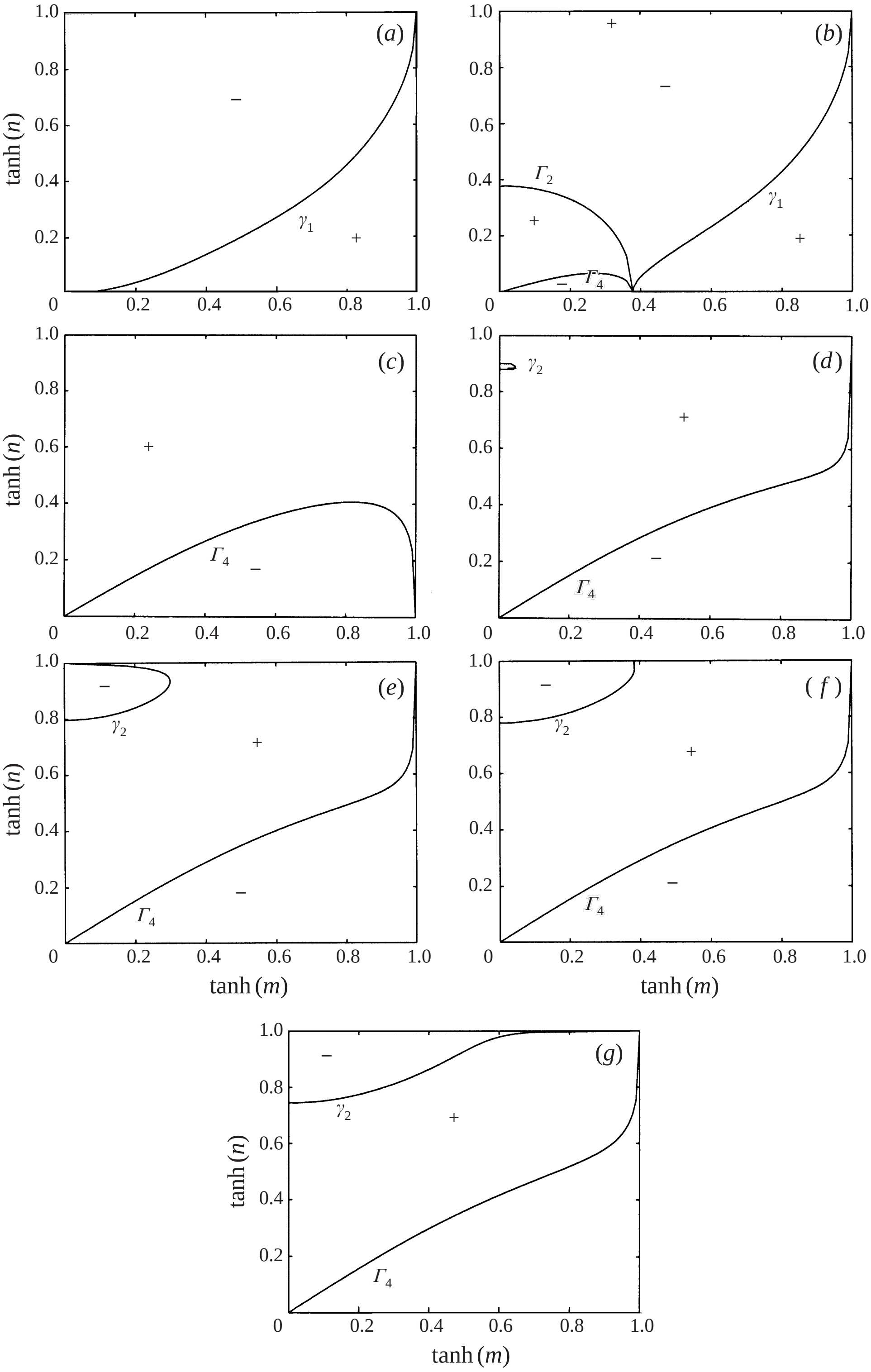

FIGURE 3. For caption see facing page.

and points in parameter space where the determinant changes sign are often associated with changes in stability. In Bridges (1998), this stability theory is extended to multi-phase patterns and quasi-periodic patterns. Preliminary results show that this theory applies to give a linear stability theory for multi-periodic patterns in shallow water – particularly short-crested waves – using the parameter structure and Jacobian matrices on parameter space.

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Appendix A. Expressions for $\nabla S(\boldsymbol{Z})$, $\mathbf{J}(\boldsymbol{u})$ and $\mathbf{K}(\boldsymbol{v})$

The matrix-valued operators appearing in (2.10) have the following expressions:

$$\mathbf{J}(\boldsymbol{u}) = \begin{bmatrix} 0 & -\boldsymbol{u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{u} & 0 & 0 & 0 & 0 & -\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{K}(\boldsymbol{v}) = \begin{bmatrix} 0 & -\boldsymbol{v} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{v} & 0 & 0 & 0 & 0 & 0 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

FIGURE 3. Sign of the coefficient $\alpha_2 - \alpha_3$ in the $[\tanh(\kappa h_0), \tanh(\ell h_0)]$-plane for a range of fixed values of the Bond number τ. The vanishing of this coefficient is associated with a secondary bifurcation of solutions. The curves γ_1 and γ_2 in this figure represent the zeros of $\alpha_2 - \alpha_3$ while Γ_2 and Γ_4 represent the poles of $\alpha_2 - \alpha_3$. The poles correspond to linear resonances. In water of infinite depth, in the absence of surface tension, $\alpha_2 - \alpha_3$ vanishes for $\overline{\kappa} = 0.893$. The values of the Bond number for each figure are (*a*) $\tau = 0$, (*b*) $\tau = 0.05$, (*c*) $\tau = 0.5$, (*d*) $\tau = 0.65$, (*e*) $\tau = 0.7$, (*f*) $\tau = 0.725$, (*g*) $\tau = 0.8$.

The gradient of $S(\boldsymbol{Z})$, defined in (2.14), with respect to the inner product (2.12) is

$$\nabla S(\boldsymbol{Z}) = \begin{pmatrix} \partial S/\partial \Phi \\ \partial S/\partial \eta \\ \partial S/\partial \gamma_1 \\ \partial S/\partial \gamma_2 \\ \partial S/\partial p \\ \partial S/\partial w_1 \\ \partial S/\partial w_2 \\ \partial S/\partial \phi \\ \partial S/\partial u \\ \partial S/\partial v \end{pmatrix} = \begin{pmatrix} -\phi_z|_{z=\eta} \\ \frac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \phi_z^2)|_{z=\eta} - g\eta + p \\ 0 \\ 0 \\ \eta \\ \sigma w_1/\sqrt{1-w_1^2-w_2^2} \\ \sigma w_2/\sqrt{1-w_1^2-w_2^2} \\ \phi_{zz} \\ u \\ v \end{pmatrix}.$$

Appendix B. Evaluation of the functionals $\mathscr{S}, \mathscr{A}_1, \mathscr{A}_2, \mathscr{R}, \mathscr{Q}_1, \mathscr{Q}_2$ in §6

Expressions for the functionals $\mathscr{S}$, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{R}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ evaluated on the Fourier expansion of the doubly periodic pattern (6.3)–(6.6) are

$$\begin{aligned}
\mathscr{S} = {} & \tfrac{1}{4} K^2 h_0 (1 - T_{10}^2)(|B_1|^2 + |B_2|^2) \\
& -\tfrac{1}{8} K^3 T_{10} \operatorname{Re}(A_1^2 \overline{B_1}^2 + A_2^2 \overline{B_2}^2) - \tfrac{1}{8} K^2 (1 + T_{10}^2) \operatorname{Re}(A_3 \overline{B_1}^2 + A_4 \overline{B_2}^2) \\
& -\tfrac{1}{2} K\, T_{10} \operatorname{Re}(\kappa^2 A_1 A_2 \overline{B_1 B_2} + \ell^2 A_1 \overline{A_2 B_1} B_2) \\
& +\tfrac{1}{4} (\ell^2 - \kappa^2 - K^2 T_{10}^2) \operatorname{Re}(A_5 \overline{B_1 B_2}) - \tfrac{1}{4} (\ell^2 - \kappa^2 + K^2 T_{10}^2) \operatorname{Re}(A_6 \overline{B_1} B_2) \\
& +\tfrac{1}{2} K^2 (1 - T_{10} T_{20}) \operatorname{Re}(A_1 B_1 \overline{B_3} + A_2 B_2 \overline{B_4}) \\
& +\tfrac{1}{2} \kappa (\kappa - K\, T_{10}\, T_{21}) \operatorname{Re}(A_1 B_2 \overline{B_5} + A_2 B_1 \overline{B_5}) \\
& +\tfrac{1}{2} \ell (\ell - K\, T_{10}\, T_{22}) \operatorname{Re}(A_1 \overline{B_2 B_6} + A_2 \overline{B_1} B_6) \\
& +K^2 h (1 - T_{20}^2)(|B_3|^2 + |B_4|^2) + \kappa^2 h (1 - T_{21}^2) |B_5|^2 + \ell^2 h (1 - T_{22}^2) |B_6|^2 \\
& +\tfrac{1}{2} \operatorname{Im}[(\kappa u_0 + \ell v_0) A_1 \overline{B_1} + (\kappa u_0 - \ell v_0) A_2 \overline{B_2}] \\
& +\tfrac{1}{16} K^2 \kappa u_0 |A_1|^2 \operatorname{Im}(A_1 \overline{B_1} + 2 A_2 \overline{B_2}) + \tfrac{1}{16} K^2 \kappa u_0 |A_2|^2 \operatorname{Im}(2 A_1 \overline{B_1} + A_2 \overline{B_2}) \\
& +\tfrac{1}{2} K\, T_{20} \operatorname{Im}[(\kappa u_0 + \ell v_0) A_1^2 \overline{B_3} + (\kappa u_0 - \ell v_0) A_2^2 \overline{B_4}] \\
& +\kappa u_0 \operatorname{Im}(A_3 \overline{B_3} + A_4 \overline{B_4} + A_5 \overline{B_5}) + \ell v_0 \operatorname{Im}(A_3 \overline{B_3} - A_4 \overline{B_4} + A_6 \overline{B_6}) \\
& +\tfrac{1}{16} K^2 \ell v_0 |A_1|^2 \operatorname{Im}(A_1 \overline{B_1} - 2 A_2 \overline{B_2}) + \tfrac{1}{16} K^2 \ell v_0 |A_2|^2 \operatorname{Im}(2 A_1 \overline{B_1} - A_2 \overline{B_2}) \\
& +\kappa^2 u_0 T_{21} \operatorname{Im}(A_1 A_2 \overline{B_5}) + \ell^2 v_0 T_{22} \operatorname{Im}(A_1 \overline{A_2 B_6}) \\
& +\tfrac{1}{4} K \kappa u_0 T_{10} \operatorname{Im}[(\overline{A_6 B_2} - \overline{A_3} B_1 - \overline{A_5} B_2) A_1 + (A_6 \overline{B_1} - \overline{A_4} B_2 - \overline{A_5} B_1) A_2] \\
& +\tfrac{1}{4} K \ell v_0 T_{10} \operatorname{Im}[(\overline{A_5} B_2 - \overline{A_3} B_1 - \overline{A_6 B_2}) A_1 + (\overline{A_4} B_2 - \overline{A_5} B_1 + A_6 \overline{B_1}) A_2] \\
& -\tfrac{1}{4} g (|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 + |A_5|^2 + |A_6|^2) + \tfrac{1}{4} \sigma K^2 (|A_1|^2 + |A_2|^2) \\
& -\tfrac{9}{64} \sigma K^4 (|A_1|^4 + |A_2|^4) - \tfrac{9}{16} \sigma (K^4 - \tfrac{8}{3} \kappa^2 \ell^2) |A_1|^2 |A_2|^2 + \sigma K^2 (|A_3|^2 + |A_4|^2) \\
& +\sigma \kappa^2 |A_5|^2 + \sigma \ell^2 |A_6|^2 + \tfrac{1}{2} ({u_0}^2 + {v_0}^2) h_0 + p h_0 + \cdots .
\end{aligned}$$

The mean flow constraints are as follows:

$$\mathscr{R} = p, \quad \mathscr{Q}_1 = u_0 h_0 + \kappa \,\mathrm{Im}\,[\widehat{\mathscr{Q}}_1] \quad \text{and} \quad \mathscr{Q}_2 = v_0 h_0 + \ell \,\mathrm{Im}\,[\widehat{\mathscr{Q}}_2],$$

where

$$\begin{aligned}
\widehat{\mathscr{Q}}_1 = {} & \tfrac{1}{2}\,(A_1\overline{B_1} + A_2\overline{B_2}) \\
& + \tfrac{1}{16} K^2\,|A_1|^2\,(A_1\overline{B_1} + 2A_2\overline{B_2}) + \tfrac{1}{16} K^2\,|A_2|^2\,(2A_1\overline{B_1} + A_2\overline{B_2}) \\
& + \tfrac{1}{2} K\; T_{20}\,(A_1^2\overline{B_3} + A_2^2\overline{B_4}) + A_3\overline{B_3} + A_4\overline{B_4} + A_5\overline{B_5} + \kappa\; T_{21}\,A_1A_2\overline{B_5} \\
& + \tfrac{1}{4} K\; T_{10}\,[A_1(\overline{A_6B_2} - \overline{A_3}B_1 - \overline{A_5}B_2) + A_2(A_6\overline{B_1} - \overline{A_4}B_2 - \overline{A_5}B_1)] + \cdots .
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathscr{Q}}_2 = {} & \tfrac{1}{2}\,(A_1\overline{B_1} - A_2\overline{B_2}) \\
& + \tfrac{1}{16} K^2\,|A_1|^2\,(A_1\overline{B_1} - 2A_2\overline{B_2}) + \tfrac{1}{16} K^2\,|A_2|^2\,(2A_1\overline{B_1} - A_2\overline{B_2}) \\
& + \tfrac{1}{2} K\; T_{20}\,(A_1^2\overline{B_3} - A_2^2\overline{B_4}) + A_3\overline{B_3} - A_4\overline{B_4} + A_6\overline{B_6} + \ell\; T_{22}\,A_1\overline{A_2B_6} \\
& + \tfrac{1}{4} K\; T_{10}\,[A_1(\overline{A_5}B_2 - \overline{A_3}B_1 - \overline{A_6B_2}) + A_2(A_6\overline{B_1} + \overline{A_4}B_2 - \overline{A_5}B_1)] + \cdots .
\end{aligned}$$

The wave constraints (components of wave-action flux) are as follows:

$$\begin{aligned}
\kappa\,\mathscr{A}_1 = {} & \tfrac{1}{2}\,\sigma\,\kappa^2\,(|A_1|^2 + |A_2|^2) + \tfrac{1}{4}\,(\kappa^2/K)\,(T_{10} + Kh[1 - T_{10}^2])\,(|B_1|^2 + |B_2|^2) \\
& + \tfrac{1}{2}\,\kappa\,u_0\,\mathrm{Im}\,(A_1\overline{B_1} + A_2\overline{B_2}) + \tfrac{1}{4}\,\sigma\,\kappa^2\,(\ell^2 - 3\kappa^2)\,|A_1|^2|A_2|^2 \\
& + \tfrac{1}{16} K^2\,\kappa\,u_0\,|A_1|^2\,\mathrm{Im}\,(A_1\overline{B_1} + 2A_2\overline{B_2}) + \tfrac{1}{16} K^2\,\kappa\,u_0\,|A_2|^2\,\mathrm{Im}\,(2A_1\overline{B_1} + A_2\overline{B_2}) \\
& + \tfrac{1}{8} K\,\kappa^2\,T_{10}\,[2\,(|A_1|^2 + |A_2|^2)\,(|B_1|^2 + |B_2|^2) - \mathrm{Re}\,(A_1^2\overline{B_1}^2 + A_2^2\overline{B_2}^2)] \\
& + \tfrac{1}{2} K\,\kappa\,u_0\,T_{20}\,\mathrm{Im}\,(A_1^2\overline{B_3} + A_2^2\overline{B_4}) + \kappa^2\,u_0\,T_{21}\,\mathrm{Im}\,(A_1A_2\overline{B_5}) \\
& + \tfrac{1}{4} K\,\kappa\,u_0\,T_{10}\,\mathrm{Im}\,[(\overline{A_6B_2} - \overline{A_3}B_1 - \overline{A_5}B_2)\,A_1 + (A_6\overline{B_1} - \overline{A_4}B_2 - \overline{A_5}B_1)A_2] \\
& - \tfrac{1}{4}\,\kappa^2\,\mathrm{Re}\,(A_3\overline{B_1}^2 + A_4\overline{B_2}^2) + \tfrac{1}{2} K\,\kappa^2\,T_{10}\,\mathrm{Re}\,(A_1\overline{A_2B_1}B_2 - A_1A_2\overline{B_1B_2}) \\
& + \tfrac{1}{2}\,\kappa^2\,\mathrm{Re}\,(A_6\overline{B_1}B_2 - A_5\overline{B_1B_2} + 2A_1B_1\overline{B_3} + 2A_2B_2\overline{B_4} + 2A_1B_2\overline{B_5} + 2A_2B_1\overline{B_5}) \\
& - \tfrac{3}{16}\,\sigma\,K^2\,\kappa^2\,(|A_1|^4 + |A_2|^4) + \kappa\,u_0\,\mathrm{Im}\,(A_3\overline{B_3} + A_4\overline{B_4} + A_5\overline{B_5}) \\
& + 2\,\sigma\,\kappa^2\,(|A_3|^2 + |A_4|^2 + |A_5|^2) + \tfrac{1}{2}\,(\kappa^2/K)\,(T_{20} + 2Kh[1 - T_{20}^2])\,(|B_3|^2 + |B_4|^2) \\
& + \tfrac{1}{2}\,\kappa\,(T_{21} + 2\kappa\,h\,[1 - T_{21}^2])\,|B_5|^2 + \cdots ,
\end{aligned}$$

$$\begin{aligned}
\ell\,\mathscr{A}_2 = {} & \tfrac{1}{2}\,\sigma\,\ell^2\,(|A_1|^2 + |A_2|^2) + \tfrac{1}{4}\,(\ell^2/K)\,(T_{10} + Kh[1 - T_{10}^2])\,(|B_1|^2 + |B_2|^2) \\
& + \tfrac{1}{2}\,\ell\,v_0\,\mathrm{Im}\,(A_1\overline{B_1} - A_2\overline{B_2}) + \tfrac{1}{4}\,\sigma\,\ell^2\,(\kappa^2 - 3\ell^2)\,|A_1|^2|A_2|^2 \\
& + \tfrac{1}{16} K^2\,\ell\,v_0\,|A_1|^2\,\mathrm{Im}\,(A_1\overline{B_1} - 2A_2\overline{B_2}) + \tfrac{1}{16} K^2\,\ell\,v_0\,|A_2|^2\,\mathrm{Im}\,(2A_1\overline{B_1} - A_2\overline{B_2}) \\
& + \tfrac{1}{8} K\,\ell^2\,T_{10}\,[2\,(|A_1|^2 + |A_2|^2)\,(|B_1|^2 + |B_2|^2) - \mathrm{Re}\,(A_1^2\overline{B_1}^2 + A_2^2\overline{B_2}^2)] \\
& + \tfrac{1}{2} K\,\ell\,v_0\,T_{20}\,\mathrm{Im}\,(A_1^2\overline{B_3} - A_2^2\overline{B_4}) + \ell^2\,v_0\,T_{22}\,\mathrm{Im}\,(A_1\overline{A_2B_6})
\end{aligned}$$

$$+\tfrac{1}{4}\,K\,\ell\,v_0\,T_{10}\,\mathrm{Im}\,[(\overline{A_5}B_2-\overline{A_3}B_1-\overline{A_6B_2})\,A_1+(\overline{A_4}B_2-\overline{A_5}B_1+A_6\overline{B_1})A_2]$$

$$-\tfrac{1}{4}\,\ell^2\,\mathrm{Re}\,(A_3\overline{B_1}^2+A_4\overline{B_2}^2)+\tfrac{1}{2}\,K\,\ell^2\,T_{10}\,\mathrm{Re}\,(A_1A_2\overline{B_1B_2}-A_1\overline{A_2B_1}B_2)$$

$$+\tfrac{1}{2}\,\ell^2\,\mathrm{Re}\,(A_5\overline{B_1B_2}-A_6\overline{B_1}B_2+2A_1B_1\overline{B_3}+2A_2B_2\overline{B_4}+2A_1\overline{B_2B_6}+2A_2\overline{B_1}B_6)$$

$$-\tfrac{3}{16}\,\sigma\,K^2\,\ell^2\,(|A_1|^4+|A_2|^4)+\ell\,v_0\,\mathrm{Im}\,(A_3\overline{B_3}-A_4\overline{B_4}+A_6\overline{B_6})$$

$$+2\,\sigma\,\ell^2\,(|A_3|^2+|A_4|^2+|A_6|^2)+\tfrac{1}{2}\,(\ell^2/K)\,(T_{20}+2Kh[1-T_{20}^2])\,(|B_3|^2+|B_4|^2)$$

$$+\tfrac{1}{2}\,\ell\,(T_{22}+2\ell\,h\,[1-T_{22}^2])\,|B_6|^2+\cdots.$$

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